5

General Engineering and Mathematics

Constants

Types of Numbers

Natural numbers

The set of *natural numbers*, {0, 1, 2, ...}, is customarily denoted by $\mathbb{N}$. Many authors do not consider 0 to be a natural number.

Integers

The set of *integers*, {0, ±1, ±2, ...}, is customarily denoted by $\mathbb{Z}$. The *positive integers* are {1, 2, 3, ...}.

Rational numbers

The set of *rational numbers*, $\left\{\frac{p}{q} \middle| p, q \in \mathbb{Z}, q \neq 0\right\}$, is customarily denoted by $\mathbb{Q}$. Two fractions $\frac{p}{q}$ and $\frac{r}{s}$ are equal if and only if $ps = qr$.

Addition of fractions is defined by $\frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs}$. Multiplication of fractions is defined by $\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$.

Real numbers

The set of *real numbers* is customarily denoted by $\mathbb{R}$. Real numbers are defined to be converging sequences of rational numbers or as decimals that might or might not repeat.

Real numbers are often divided into two subsets. One subset, the *algebraic numbers*, are real numbers which solve a polynomial equation in one variable with integer coefficients. For example: $\frac{1}{\sqrt{2}}$ is an algebraic number because it solves the polynomial equation $2x^2 - 1 = 0$; and all rational numbers are algebraic. Real numbers that are not algebraic numbers are called *transcendental numbers*. Examples of transcendental numbers include π and e.

Complex numbers

The set of *complex numbers* is customarily denoted by $\mathbb{C}$. They are numbers of the form $a + bi$, where $i^2 = -1$, and a and b are real numbers.

Operation	Computation	Result
addition	$(a + bi) + (c + di)$	$(a + c) + i(b + d)$
multiplication	$(a + bi)(c + di)$	$(ac - bd) + (ad + bc)i$
reciprocal	$\frac{1}{a+bi}$	$\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$
complex conjugate	$z = a + bi$	$\bar{z} = a - bi$

Properties include: $\overline{z + w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z}\bar{w}$.

From Bolinger, K., Glasser, M.L., Gross, R., and Sloane, N.J.A., Analysis, in *CRC Standard Mathematical Tables and Formulae*, 31st ed., Zwillinger, D., Ed., CRC Press, Boca Raton, FL, 2003, p. 3.

Decimal Multiples and Prefixes

The prefix names and symbols below re taken from Conference Générale des Poids et Mesures, 1991. The common names are for the U.S.

Factor	Prefix	Symbol	Common Name
$10^{(10^{100})}$			googolplex
10^{100}			googol
10^{24}	yotta	Y	heptillion
10^{21}	zetta	Z	hexillion
10^{18}	exa	E	quintillion
10^{15}	peta	P	quadrillion
10^{12}	tera	T	trillion
10^{9}	giga	G	billion
10^{6}	mega	M	million
10^{3}	kilo	k	thousand
10^{2}	hecto	H	hundred
10^{1}	deka	da	ten
10^{-1}	deci	d	tenth
10^{-2}	centi	c	hundreth
10^{-3}	milli	m	thousandth
10^{-6}	micro	μ (Greek mu)	millionth
10^{-9}	nano	n	billionth
10^{-12}	pico	p	trillionth
10^{-15}	femto	f	quadrillionth
10^{-18}	atto	a	quintillionth
10^{-21}	zepto	z	hexillionth
10^{-24}	yocto	y	heptillionth

From Bolinger, K., Glasser, M.L., Gross, R., and Sloane, N.J.A., Analysis, in *CRC Standard Mathematical Tables and Formulae*, 31st ed., Zwillinger, D., Ed., CRC Press, Boca Raton, FL, 2003, p. 6.

Powers of 10 In Hexadecimal Scale

n	10^n	10^{-n}
0	1_{16}	1_{16}
1	A_{16}	$0.1999999999999999999\ldots_{16}$
2	64_{16}	$0.028F5C28F5C28F5C28F5\ldots_{16}$
3	$3E8_{16}$	$0.004189374BC6A7EF9DB2\ldots_{16}$
4	2710_{16}	$0.00068DB8BAC710CB295E\ldots_{16}$
5	$186A0_{16}$	$0.000A7C5AC471B478412\ldots_{16}$
6	$F4240_{16}$	$0.000010C6F7A0B5ED8D36\ldots_{16}$
7	989680_{16}	$0.000001AD7F29ABCAF485\ldots_{16}$
8	$5F5E100_{16}$	$0.0000002AF31DC4611873\ldots_{16}$
9	$3B9ACA00_{16}$	$0.000000044B82FA09B5A5\ldots_{16}$
10	$2540BE400_{16}$	$0.000000006DF37F675EF6\ldots_{16}$
11	$174876E800_{16}$	$0.000000000AFEBFF0BCB2\ldots_{16}$
12	$E8D43A51000_{16}$	$0.00000000119799812DE\ldots_{16}$
13	$9184E72A000_{16}$	$0.00000000001C25C26849\ldots_{16}$
14	$5AF3107A4000_{16}$	$0.000000000002D09370D4\ldots_{16}$
15	$38D7EA4C68000_{16}$	$0.000000000000480EBE7B\ldots_{16}$
16	$2386F26FC10000_{16}$	$0.0000000000000734ACA5\ldots_{16}$

From Bolinger, K., Glasser, M.L., Gross, R., and Sloane, N.J.A., Analysis, in *CRC Standard Mathematical Tables and Formulae*, 31st ed., Zwillinger, D., Ed., CRC Press, Boca Raton, FL, 2003, p. 13.

Factorials

For non-negative integers n, the factorial of n, denoted $n!$, is the product of all positive integers less than or equal to n; $n! = n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1$. If n is a negative integer ($n = -1, -2, \ldots$) then $n! = \pm\infty$. Note that, since the empty product is 1, it follows that $0! = 1$. The generalization of the factorial function to non-integer arguments is the gamma function. When n is an integer, $\Gamma(n) = (n-1)!$.

The double factorial of n, denoted $n!!$, is the product of every other integer: $n!! = n\cdot(n-2)\cdot(n-4)\cdots$, where the last element in the product is either 2 or 1, depending on whether n is even or odd. The *shifted factorial* (also called the *rising factorial and Pochhammer's symbol*) is denoted by $(a)_n$ (sometimes $a^{\bar{n}}$) and is defined as

$$(a)_n = \underbrace{a\cdot(a+1)\cdot(a+2)\cdots(a+n-1)}_{n\ terms} = \frac{(a+n-1)!}{(a-1)!} = \frac{\Gamma(a+n)}{\Gamma(a)}$$

Approximations to $n!$ for large n include Stirling's formula

$$n! \approx \sqrt{2\pi e}\left(\frac{n}{e}\right)^{n+\frac{1}{2}}$$

and Burnsides's formula

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}$$

n	$n!$	$\log_{10} n!$	$n!!$	$\log_{10} n!!$
0	1	0.00000	1	0.00000
1	1	0.00000	1	0.00000
2	2	0.30103	2	0.30103
3	6	0.77815	3	0.47712
4	24	1.38021	8	0.90309
5	120	2.07918	15	1.17609
6	720	2.85733	48	1.68124
7	5040	3.70243	105	2.02119
8	40320	4.60552	384	2.58433
9	3.6288×10^{5}	5.55976	945	2.97543
10	3.6288×10^{6}	6.55976	3840	3.58433
11	3.9917×10^{7}	7.60116	10395	4.01682
12	4.7900×10^{8}	8.68034	46080	4.66351
13	6.2270×10^{9}	9.79428	1.3514×10^{5}	5.13077
14	8.7178×10^{10}	10.94041	6.4512×10^{5}	5.80964
15	1.3077×10^{12}	12.11650	2.0270×10^{6}	6.30686
16	2.0923×10^{13}	13.32062	1.0322×10^{7}	7.01376
17	3.5569×10^{14}	14.55107	3.4459×10^{7}	7.53731
18	6.4024×10^{15}	15.80634	1.8579×10^{8}	8.26903
19	1.2165×10^{17}	17.08509	6.5473×10^{8}	8.81606
20	2.4329×10^{18}	18.38612	3.7159×10^{9}	9.57006
21	5.1091×10^{19}	19.70834	1.3749×10^{10}	10.13828
22	1.1240×10^{21}	21.05077	8.1750×10^{10}	10.91249
23	2.5852×10^{22}	22.41249	3.1623×10^{11}	11.50001
24	6.2045×10^{23}	23.79271	1.9620×10^{12}	12.29270
25	1.5511×10^{25}	25.19065	7.9059×10^{12}	12.89795
30	2.6525×10^{32}	32.42366	4.2850×10^{16}	16.63195
40	8.1592×10^{47}	47.91165	2.5511×10^{24}	24.40672
50	3.0414×10^{64}	64.48307	5.2047×10^{32}	32.71640

Factorials (continued)

60	8.3210×10^{81}	81.92017	2.8481×10^{41}	41.45456
70	1.1979×10^{100}	100.07841	3.5504×10^{50}	50.55028
80	7.1569×10^{118}	118.85473	8.9711×10^{59}	59.95284
90	1.4857×10^{138}	138.17194	4.2088×10^{69}	69.62416
100	9.3326×10^{157}	157.97000	3.4243×10^{79}	79.53457
110	1.5882×10^{178}	178.20092	4.5744×10^{89}	89.66033
120	6.6895×10^{198}	198.82539	9.5934×10^{99}	99.98197
130	6.4669×10^{219}	219.81069	3.0428×10^{110}	110.48328
140	1.3462×10^{241}	241.12911	1.4141×10^{121}	121.15050
150	5.7134×10^{262}	262.75689	9.3726×10^{131}	131.97186
500	1.2201×10^{1134}	1134.0864	5.8490×10^{567}	567.76709
1000	4.0239×10^{2567}	2567.6046	3.9940×10^{1284}	1284.6014

From Bolinger, K., Glasser, M.L., Gross, R., and Sloane, N.J.A., Analysis, in *CRC Standard Mathematical Tables and Formulae*, 31st ed., Zwillinger, D., Ed., CRC Press, Boca Raton, FL, 2003, pp. 17–18.

Prime Numbers

1. A *prime number* is a positive integer greater than 1 with no positive, integral divisors other than 1 and itself. There are infinitely many prime numbers, 2, 3, 5, 7,…. The sum of the reciprocals of the prime numbers diverges: $\sum_n \frac{1}{pn} = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = \infty$.
2. *Twin primes* are prime numbers that differ by two: (3, 5), (5, 7), (11, 13), (17, 19),…. It is not known whether there are infinitely many twin primes. The sum of the reciprocals of the twin primes converges; the value
$$B = \left(\frac{1}{3} + \frac{1}{5}\right) + \left(\frac{1}{5} + \frac{1}{7}\right) + \left(\frac{1}{11} + \frac{1}{13}\right) + \cdots + \left(\frac{1}{p} + \frac{1}{p+2}\right) + \cdots$$
known as Brun's constant is approximately $B \approx 1.90216054$.
3. For every integer $n \geq 2$, the numbers $\{n! + 2, n! + 3, \ldots, n! + n\}$ are a sequence of $n - 1$ consecutive composite (i.e., not prime) numbers.
4. *Dirichlet's theorem on primes in arithmetic progressions*: Let a and b be relatively prime positive integers. Then the arithmetic progression $an + b$ (for $n = 1, 2, \ldots$) contains infinitely many primes.
5. *Goldbach conjecture*: every even number is the sum of two prime numbers.
6. The function $\pi(x)$ represents the number of primes less than x. The prime number theorem states that $\pi(x) \sim x/\log x$ as $x \to \infty$. The exact number of primes less than a given number is:

x	100	1000	10,000	10^5	10^6	10^7	10^8
$\pi(x)$	25	168	1,229	9,592	78,498	664,579	5,761,455

x	10^{10}	10^{15}	10^{21}
$\pi(x)$	455,052,511	29,844,570,422,669	21,127,269,486,018,731,928

From Driscoll, P.J., Gross, R., Michaels, J., Nelsen, R.B., and Wilson, B., Algebra, in *CRC Standard Mathematical Tables and Formulae*, 31st ed., Zwillinger, D., Ed., CRC Press, Boca Raton, FL, 2003, p. 103.

Reliability

1. The *reliability* of a product is the probability that the product will function within specified limits for at least a specified period of time.
2. A *series system* is one in which the entire system will fail if any of its components fail.
3. A *parallel system* is one in which the entire system will fail only if all of its components fail.
4. Let R_i denote the reliability of the i^{th} component.
5. Let R_s denote the reliability of a series system.
6. Let R_p denote the reliability of a series system.

The *product law of reliabilities* states

$$R_s = \prod_{i=1}^{n} R_i$$

The *product law of unreliabilities* states

$$R_p = 1 - \prod_{i=1}^{n} \left(1 - R_i\right)$$

From Mascagni, M., Rinaman, W.C., Sousa, M., and Strauss, M.T., Probability and statistics, in *CRC Standard Mathematical Tables and Formulae*, 31st ed., Zwillinger, D., Ed., CRC Press, Boca Raton, FL, 2003, p. 653.

Conversion: Metric to English

Multiply	By	To Obtain
centimeters	0.3937008	inches
cubic meters	1.307951	cubic yards
cubic meters	35.31467	cubic feet
grams	0.03527396	ounces
kilograms	2.204623	pounds
kilometers	0.6213712	miles
liters	0.2641721	gallons (US)
meters	1.093613	yards
meters	3.280840	feet
milliliters	0.03381402	fluid ounces
milliliters	0.06102374	cubic inches
square centimeters	0.1550003	square inches
square meters	1.195990	square yards
square meters	10.76391	square feet

From Gross, R., Katz, V.J., and Strauss, M.T., Miscellaneous, in *CRC Standard Mathematical Tables and Formulae*, 31st ed., Zwillinger, D., Ed., CRC Press, Boca Raton, FL, 2003, p. 796.

Conversion: English to Metric

Multiply	By	To Obtain
cubic feet	0.02831685	cubic meters
cubic inches	16.38706	milliliters
cubic yards	0.7645549	cubic meters
feet	0.3048000	meters
fluid ounces	29.57353	milliliters
gallons (US)	3.785412	liters
inches	2.540000	centimeters
miles	1.609344	kilometers
mils	25.4	micrometers

Conversion: English to Metric (continued)

Multiply	By	To Obtain
ounces	28.34952	grams
pounds	0.4535924	kilograms
square feet	0.09290304	square meters
square inches	6.451600	square centimeters
square yards	0.8361274	square meters
yards	0.9144000	meters

From Gross, R., Katz, V.J., and Strauss, M.T., Miscellaneous, in *CRC Standard Mathematical Tables and Formulae*, 31st ed., Zwillinger, D., Ed., CRC Press, Boca Raton, FL, 2003, p. 797.

Interpretations of Powers of 10

10^{-15}	the radius of the hydrogen nucleus (a proton) in meters
10^{-11}	the likelihood of being dealt 13 top honors in bridge
10^{-10}	the radius of a hydrogen atom in meters
10^{-9}	the number of seconds it takes light to travel one foot
10^{-6}	the likelihood of being dealt a royal flush in poker
10^{0}	the density of water is 1 gram per milliliter
10^{1}	the number of fingers that people have
10^{2}	the number of sable elements in the periodic table
10^{5}	the number of hairs on a human scalp
10^{6}	the number of possible chess board positions after 4 moves
10^{7}	the number of seconds in a year
10^{8}	the speed of light in meters per second
10^{9}	the number of heartbeats in a lifetime for most mammals
10^{10}	the number of people on the earth
10^{15}	the surface area of the earth in square meters
10^{16}	the age of the universe in seconds
10^{18}	the volume of water in the earth's oceans in cubic meters
10^{19}	the number of possible positions of Rubik's cube
10^{21}	the volume of the earth in cubic meters
10^{24}	the number of grains of sand in the Sahara desert
10^{28}	the mass of the earth in grams
10^{33}	the mass of the solar system in grams
10^{50}	the number of atoms in the earth
10^{78}	the volume of the universe in cubic meters

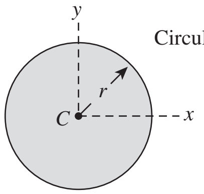
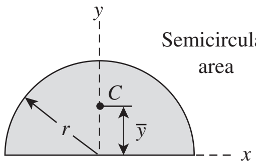
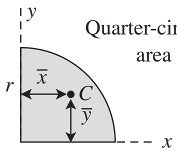
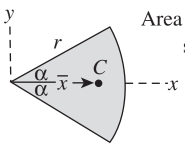
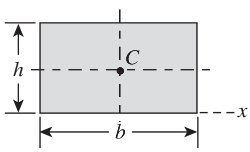
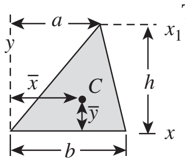
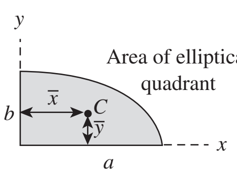
From Gross, R., Katz, V.J., and Strauss, M.T., Miscellaneous, in *CRC Standard Mathematical Tables and Formulae*, 31st ed., Zwillinger, D., Ed., CRC Press, Boca Raton, FL, 2003, pp. 798–799.

Typical Values for Coefficients of Static Friction

Materials	μ_s
Metal on ice	0.03–0.05
Wood on wood	0.30–0.70
Leather on wood	0.20–0.50
Leather on metal	0.30–0.60
Aluminum on aluminum	1.10–1.70

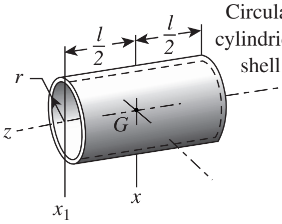
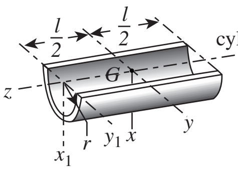
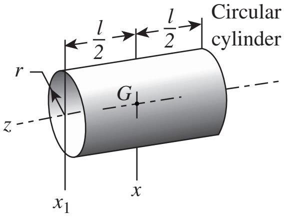
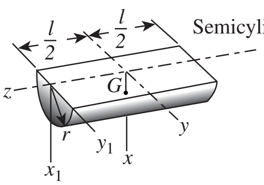
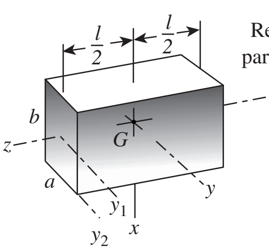
From Hibbeler, R.C., Force-system resultants and equilibrium, in *the Engineering Handbook,* Dorf, R.C., Ed., CRC Press, Boca Raton, FL, 1996, p. 8.

Properties of Plane Areas

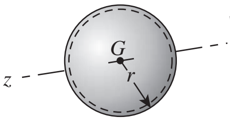
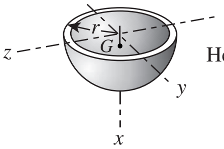
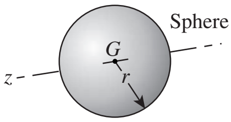
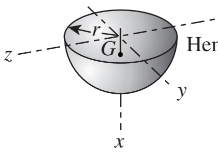
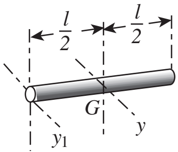
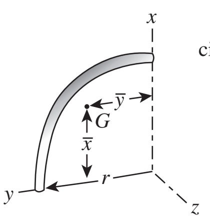
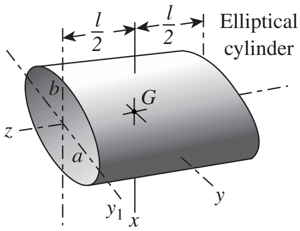
Figure	Centroid	Area Moments of Inertia
Circular area 	—	$I_x = I_y = \frac{\pi r^4}{4}$ $I_z = \frac{\pi r^4}{2}$
Semicircular area 	$\bar{y} = \frac{4r}{3\pi}$	$I_x = I_y = \frac{\pi r^4}{8}$ $\bar{I}_x = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4$ $I_z = \frac{\pi r^4}{4}$
Quarter-circular area 	$\bar{x} = \bar{y} = \frac{4r}{3\pi}$	$I_x = I_y = \frac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4$ $I_z = \frac{\pi r^4}{8}$
Area of circular sector 	$\bar{x} = \frac{2}{3}\frac{r \sin\alpha}{\alpha}$	$I_x = \frac{r^4}{4}\left(\alpha - \frac{1}{2}\sin 2\alpha\right)$ $I_y = \frac{r^4}{4}\left(\alpha + \frac{1}{2}\sin 2\alpha\right)$ $I_z = \frac{1}{2} r^4 \alpha$
Rectangular area 	—	$I_x = \frac{bh^3}{3}$ $\bar{I}_x = \frac{bh^3}{12}$ $\bar{I}_z = \frac{bh}{12}(b^2 + h^2)$
Triangular area 	$\bar{x} = \frac{a+b}{3}$ $\bar{y} = \frac{h}{3}$	$I_x = \frac{bh^3}{12}$ $I_x = \frac{bh^3}{36}$ $I_{x_1} = \frac{bh^3}{4}$
Area of elliptical quadrant 	$\bar{x} = \frac{4a}{3\pi}$ $\bar{y} = \frac{4b}{3\pi}$	$I_x = \frac{\pi ab^3}{16}$, $\bar{I}_x = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) ab^3$ $I_y = \frac{\pi a^3 b}{16}$, $\bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) a^3 b$ $I_z = \frac{\pi ab}{16}(a^2 + b^2)$

From Meriam, J.L., Moments of inertia, in *The Engineering Handbook,* Dorf, R.C., Ed., CRC Press, Boca Raton, FL, 1996, p. 30. Originally from Meriam, J. L. and Kraige, L. G. 1992. *Engineering Mechanics*, 3rd ed. John Wiley & Sons, New York.

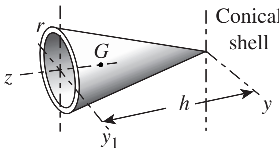
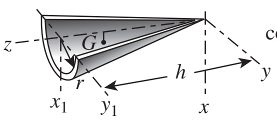
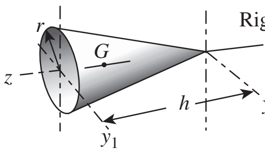
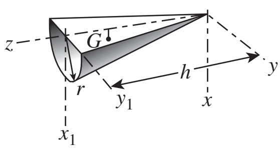
Moments of Inertia of Homogeneous Solids (m = Mass of Body Shown)

Body	Mass Center	Mass Moments of Inertia
 Circular cylindrical shell	—	$I_{xx} = \frac{1}{2}mr^2 + \frac{1}{12}ml^2$ $I_{x_1x_1} = \frac{1}{2}mr^2 + \frac{1}{3}ml^2$ $I_{zz} = mr^2$
 Half cylindrical shell	$\bar{x} = \frac{2r}{\pi}$	$I_{xx} = I_{yy}$ $= \frac{1}{2}mr^2 + \frac{1}{12}ml^2$ $I_{x_1x_1} = I_{y_1y_1}$ $= \frac{1}{2}mr^2 + \frac{1}{3}ml^2$ $I_{zz} = mr^2$ $\bar{I}_{zz} = \left(1 - \frac{4}{\pi^2}\right)mr^2$
 Circular cylinder	—	$I_{xx} = \frac{1}{4}mr^2 + \frac{1}{12}ml^2$ $I_{x_1x_1} = \frac{1}{4}mr^2 + \frac{1}{3}ml^2$ $I_{zz} = \frac{1}{2}mr^2$
 Semicylinder	$\bar{x} = \frac{4r}{3\pi}$	$I_{xx} = I_{yy}$ $= \frac{1}{4}mr^2 + \frac{1}{12}ml^2$ $I_{x_1x_1} = I_{y_1y_1}$ $= \frac{1}{4}mr^2 + \frac{1}{3}ml^2$ $I_{zz} = \frac{1}{2}mr^2$ $\bar{I}_{zz} = \left(\frac{1}{2} - \frac{16}{9\pi^2}\right)mr^2$
 Rectangular parallelepiped	—	$I_{xx} = \frac{1}{12}m(a^2 + l^2)$ $I_{yy} = \frac{1}{12}m(b^2 + l^2)$ $I_{zz} = \frac{1}{12}m(a^2 + b^2)$ $I_{y_1y_1} = \frac{1}{12}mb^2 + \frac{1}{3}ml^2$ $I_{y_2y_2} = \frac{1}{3}m(b^2 + l^2)$

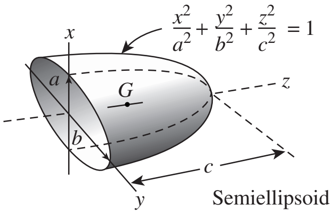
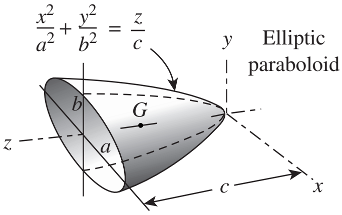
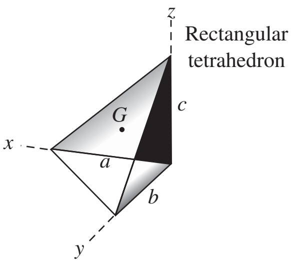
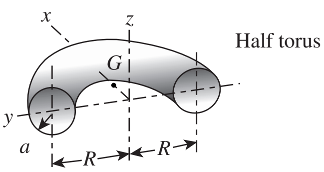
Moments of Inertia of Homogeneous Solids (m = Mass of Body Shown) (continued)

Body	Mass Center	Mass Moments of Inertia
 Spherical shell	—	$I_{zz} = \frac{2}{3}mr^2$
 Hemispherical shell	$\bar{x} = \frac{r}{2}$	$I_{xx} = I_{yy} = I_{zz} = \frac{2}{3}mr^2$ $\bar{I}_{yy} = \bar{I}_{zz} = \frac{5}{12}mr^2$
 Sphere	—	$I_{zz} = \frac{2}{5}mr^2$
 Hemisphere	$\bar{x} = \frac{3r}{8}$	$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}mr^2$ $\bar{I}_{yy} = \bar{I}_{zz} = \frac{83}{320}mr^2$
 Uniform slender rod	—	$I_{yy} = \frac{1}{12}ml^2$ $I_{y_1y_1} = \frac{1}{3}ml^2$
 Quarter-circular rod	$\bar{x} = \bar{y}$ $= \frac{2r}{\pi}$	$I_{xx} = I_{yy} = \frac{1}{2}mr^2$ $I_{zz} = mr^2$
 Elliptical cylinder	—	$I_{xx} = \frac{1}{4}ma^2 + \frac{1}{12}ml^2$ $I_{yy} = \frac{1}{4}mb^2 + \frac{1}{12}ml^2$ $I_{zz} = \frac{1}{4}m(a^2 + b^2)$ $I_{y_1y_1} = \frac{1}{4}mb^2 + \frac{1}{3}ml^2$

Moments of Inertia of Homogeneous Solids (m = Mass of Body Shown) (continued)

Body	Mass Center	Mass Moments of Inertia
	$\bar{z} = \frac{2h}{3}$	$I_{yy} = \frac{1}{4}mr^2 + \frac{1}{2}mh^2$ $I_{y_1y_1} = \frac{1}{4}mr^2 + \frac{1}{6}mh^2$ $I_{zz} = \frac{1}{2}mr^2$ $\bar{I}_{yy} = \frac{1}{4}mr^2 + \frac{1}{18}mh^2$
	$\bar{x} = \frac{4r}{3\pi}$ $\bar{z} = \frac{2h}{3}$	$I_{xx} = I_{yy}$ $= \frac{1}{4}mr^2 + \frac{1}{2}mh^2$ $I_{x_1x_1} = I_{y_1y_1}$ $= \frac{1}{4}mr^2 + \frac{1}{6}mh^2$ $I_{zz} = \frac{1}{2}mr^2$ $\bar{I}_{zz} = \left(\frac{1}{2} - \frac{16}{9\pi^2}\right)mr^2$
	$\bar{z} = \frac{3h}{4}$	$I_{yy} = \frac{3}{20}mr^2 + \frac{3}{5}mh^2$ $I_{y_1y_1} = \frac{3}{20}mr^2 + \frac{1}{10}mh^2$ $I_{zz} = \frac{3}{10}mr^2$ $\bar{I}_{yy} = \frac{3}{20}mr^2 + \frac{3}{80}mh^2$
	$\bar{x} = \frac{r}{\pi}$ $\bar{z} = \frac{3h}{4}$	$I_{xx} = I_{yy}$ $= \frac{3}{20}mr^2 + \frac{3}{5}mh^2$ $I_{x_1x_1} = I_{y_1y_1}$ $= \frac{3}{20}mr^2 + \frac{1}{10}mh^2$ $I_{zz} = \frac{3}{10}mr^2$ $\bar{I}_{zz} = \left(\frac{3}{10} - \frac{1}{\pi^2}\right)mr^2$

Moments of Inertia of Homogeneous Solids (m = Mass of Body Shown) (continued)

Body	Mass Center	Mass Moments of Inertia
	$\bar{z}=\frac{3c}{8}$	$I_{xx}=\frac{1}{5}m(b^2+c^2)$ $I_{yy}=\frac{1}{5}m(a^2+c^2)$ $I_{zz}=\frac{1}{5}m(a^2+c^2)$ $\bar{I}_{xx}=\frac{1}{5}m\left(b^2+\frac{19}{64}c^2\right)$ $\bar{I}_{yy}=\frac{1}{5}m\left(a^2+\frac{19}{64}c^2\right)$
	$\bar{z}=\frac{2c}{3}$	$I_{xx}=\frac{1}{6}mb^2+\frac{1}{2}mc^2$ $I_{yy}=\frac{1}{6}ma^2+\frac{1}{2}mc^2$ $I_{zz}=\frac{1}{6}m(a^2+b^2)$ $\bar{I}_{xx}=\frac{1}{6}m\left(b^2+\frac{1}{3}c^2\right)$ $\bar{I}_{yy}=\frac{1}{6}m\left(a^2+\frac{1}{3}c^2\right)$
	$\bar{x}=\frac{a}{4}$ $\bar{y}=\frac{b}{4}$ $\bar{z}=\frac{c}{4}$	$I_{xx}=\frac{1}{10}m(b^2+c^2)$ $I_{yy}=\frac{1}{10}m(a^2+c^2)$ $I_{zz}=\frac{1}{10}m(a^2+b^2)$ $\bar{I}_{xx}=\frac{3}{80}m(b^2+c^2)$ $\bar{I}_{yy}=\frac{3}{80}m(a^2+c^2)$ $\bar{I}_{zz}=\frac{3}{80}m(a^2+b^2)$
	$\bar{x}=\frac{a^2+4R^2}{2\pi R}$	$I_{xx}=I_{yy}=\frac{1}{2}mR^2+\frac{5}{8}ma^2$ $I_{zz}=mR^2+\frac{3}{4}ma^2$

From Meriam, J.L., Moments of inertia, in *The Engineering Handbook,* Dorf, R.C., Ed., CRC Press, Boca Raton, FL, 1996, p. 35–38.

Dynamic Viscosity of Liquids (μ) (mPa · s)

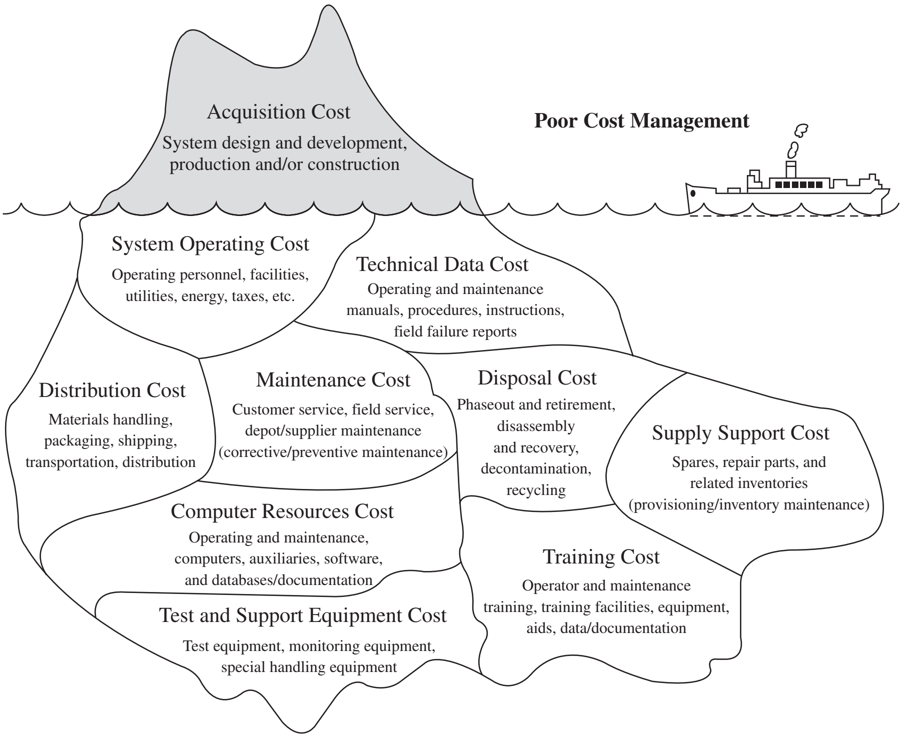
Liquid	−25°C	0°C	25°C	50°C	75°C	100°C
Water		1.793	0.890	0.547	0.378	
Mercury			1.526	1.402	1.312	
Methanol	1.258	0.793	0.544			
Isobutyl acetate			0.676	0.493	0.370	0.286
Toluene	1.165	0.778	0.560	0.424	0.333	0.270
Styrene		1.050	0.695	0.507	0.390	0.310
Acetic acid			1.056	0.786	0.599	0.464
Ethanol	3.262	1.786	1.074	0.694	0.476	
Ethylene glycol			16.1	6.554	3.340	1.975

From Braun, E.R. and Wang, P.-L., Boundary layers, in *The Engineering Handbook,* Dorf, R.C., Ed., CRC Press, Boca Raton, FL, 1996, p. 401.

Resistor Color Code

Color	First Band,[a] Significant Figure	Second Band, Significant Figure	Third Band, Multiplier	Fourth Band,[b] Tolerance (%)	Fifth Band,[b] Failure Rate (%/1000 h)
Black	0	0	1	—	—
Brown	1	1	10	—	1
Red	2	2	10^2	—	0.1
Orange	3	3	10^3	—	0.01
Yellow	4	4	10^4	—	0.001
Green	5	5	10^5	—	—
Blue	6	6	10^6	—	—
Violet	7	7	10^7	—	—
Gray	8	8	10^8	—	—
White	9	9	10^9	—	—
Silver	—	—	0.01	10	—
Gold	—	—	0.1	5	—
None	—	—	—	20	—

[a] The first band is the one closest to one end of the resistor. A first band wider than the others indicates a wire-wound resistor.
[b] Certain MIL parts.
From Domingoes, H., Passive components, in *The Engineering Handbook,* Dorf, R.C., Ed., CRC Press, Boca Raton, FL, 1996, p. 1113.



The problem of total cost visibility. From Fabrycky, W.J. and Blanchard, B.S., Life-cycle costing, in *The Engineering Handbook,* Dorf, R.C., Ed., CRC Press, Boca Raton, FL, 1996, p. 1969.

Trigonometry

Triangles

In any triangle (in a plane) with sides *a*, *b*, and *c* and corresponding opposite angles *A*, *B*, and *C*,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \text{(Law of sines)}$$

$$a^2 = b^2 + c^2 - 2cb\cos A \qquad \text{(Law of cosines)}$$

$$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} \qquad \text{(Law of tangents)}$$

$$\sin\frac{1}{2}A = \sqrt{\frac{(s-b)}{bc}} \quad \text{where } s = \frac{1}{2}(a+b+c)$$

$$\cos\frac{1}{2}A = \sqrt{\frac{s(s-a)(s-c)}{bc}}$$

$$\tan\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\text{Area} = \frac{1}{2}bc\ \sin A$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

If the vertices have coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), the area is the *absolute value* of the expression

Trigonometry (continued)

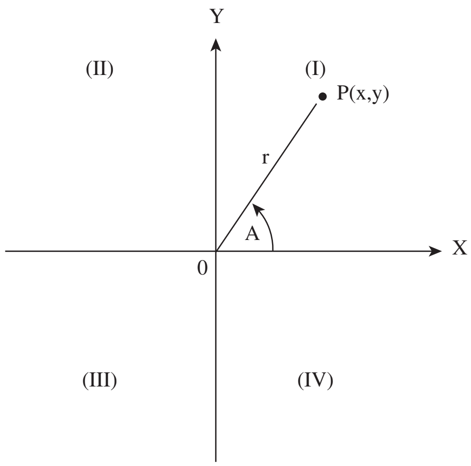
$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Trigonometric Functions of an Angle

With reference to the following figure, $P(x, y)$ is a point in any one of the four quadrants and A is an angle whose initial side is coincident with the positive x axis and whose terminal side contains the point $P(x, y)$. The distance from the origin $P(x, y)$ is denoted by r and is positive. The trigonometric functions of the angle A are defined as:

$$\sin A = \text{sine } A = y/r$$
$$\cos A = \text{cosine } A = x/r$$
$$\tan A = \text{tangent } A = y/x$$
$$\text{ctn } A = \text{cotangent } A = x/y$$
$$\sec A = \text{secant } A = r/x$$
$$\csc A = \text{cosecant } A = r/y$$

Angles are measured in degrees or radians; 180° = π radians; 1 radian = 180/π degrees.



The trigonometric point. Angle A is taken to be positive when the rotation is counterclockwise and negative when the rotation is clockwise. The plane is divided into quadrants as shown.

The trigonometric functions of 0°, 30°, 45°, and integer multiples of these are directly computed.

	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	−1
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	−1	$-\frac{\sqrt{3}}{3}$	0
ctn	∞	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	−1	$-\sqrt{3}$	∞
sec	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	∞	−2	$-\sqrt{2}$	$-\frac{2\sqrt{3}}{3}$	−1
csc	∞	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	∞

Trigonometry (continued)

Trigonometric Identities

$$\sin A = \frac{1}{\csc A}$$

$$\cos A = \frac{1}{\sec A}$$

$$\tan A = \frac{1}{\operatorname{ctn} A} = \frac{\sin A}{\cos A}$$

$$\csc A = \frac{1}{\sin A}$$

$$\sec A = \frac{1}{\cos A}$$

$$\operatorname{ctn} A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}$$

$$\sin^2 A + \cos^2 A = 1$$

$$1 + \tan^2 A = \sec^2 A$$

$$1 + \operatorname{ctn}^2 A = \csc^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\sin nA = 2 \sin(n-1)A \cos A - \sin(n-2)A$$

$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\cos nA = 2\cos(n-1)A \cos A - \cos(n-2)A$$

$$\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$$

$$\sin A - \sin B = 2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)$$

$$\cos A + \cos B = 2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$$

$$\cos A - \cos B = -2 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)$$

$$\tan A \pm \tan B = \frac{\sin(A \pm B)}{\cos A \cos B}$$

$$\operatorname{ctn} A \pm \operatorname{ctn} B = \pm \frac{\sin(A \pm B)}{\sin A \sin B}$$

$$\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$$

Trigonometry (continued)

$$\cos A \cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$$

$$\sin A \cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$$

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\tan\frac{A}{2} = \frac{1-\cos A}{\sin A} = \frac{\sin A}{1+\cos A} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

$$\sin^2 A = \frac{1}{2}(1-\cos 2A)$$

$$\cos^2 A = \frac{1}{2}(1+\cos 2A)$$

$$\sin^3 A = \frac{1}{4}(3\sin A - \sin 3A)$$

$$\cos^3 A = \frac{1}{4}(\cos 3A + 3\cos A)$$

$$\sin ix = \frac{1}{2}i\left(e^x - e^{-x}\right) = i\sinh x$$

$$\cos ix = \frac{1}{2}\left(e^x + e^{-x}\right) = \cosh x$$

$$\tan ix = \frac{i\left(e^x - e^{-x}\right)}{e^x + e^{-x}} = i\tanh x$$

$$e^{x+iy} = e^x(\cos y + i\sin y)$$

$$(\cos x \pm i\sin x)^n = \cos nx \pm i\sin nx$$

Inverse Trigonometric Functions

The inverse trigonometric functions are multiple valued, and this should be taken into account in the use of the following formulas.

$$\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$$

$$= \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \operatorname{ctn}^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$= \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \csc^{-1}\frac{1}{x}$$

$$= -\sin^{-1}(-x)$$

$$\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$$

$$= \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \operatorname{ctn}^{-1}\frac{x}{\sqrt{1-x^2}}$$

$$= \sec^{-1}\frac{1}{x} = \csc^{-1}\frac{1}{\sqrt{1-x^2}}$$

$$= \pi - \cos^{-1}(-x)$$

Trigonometry (continued)

$$\tan^{-1} x = \text{ctn}^{-1}\frac{1}{x}$$
$$= \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}}$$
$$= \sec^{-1}\sqrt{1+x^2} = \csc^{-1}\frac{\sqrt{1+x^2}}{x}$$
$$= -\tan^{-1}(-x)$$

From Dorf, R.C., Ed., *The Engineering Handbook,* CRC Press, Boca Raton, FL, 1996, pp. 2037–2041.

Series

Bernoulli and Euler Numbers

A set of numbers, B_1, B_3, . . . , B_{2n-1} (Bernoulli numbers) and B_2, B_4, . . . , B_{2n} (Euler numbers), appears in the series expansions of many functions. A partial listing follows; these are computed from the following equations:

$$B_{2n} - \frac{2n(2n-1)}{2!}B_{2n-2} + \frac{2n(2n-1)(2n-2)(2n-3)}{4!}B_{2n-4} - \cdots + (-1)^n = 0$$

and

$$\frac{2^{2n}(2^{2n}-1)}{2n}B_{2n-1} = (2n-1)B_{2n-2} - \frac{(2n-1)(2n-2)(2n-3)}{3!}B_{2n-4} + \cdots + (-1)^{n-1}$$

$B_1 = 1/6$	$B_2 = 1$
$B_3 = 1/30$	$B_4 = 5$
$B_5 = 1/42$	$B_6 = 61$
$B_7 = 1/30$	$B_8 = 1385$
$B_9 = 5/66$	$B_{10} = 50,521$
$B_{11} = 691/2730$	$B_{12} = 2,702,765$
$B_{13} = 7/6$	$B_{14} = 199,360,981$
$\vdots$	$\vdots$

Series of Functions

In the following, the interval of convergence is indicated; otherwise it is all x. Logarithms are to the base e. Bernoulli and Euler numbers (B_{2n-1} and B_{2n}) appear in certain expressions.

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots + \frac{n!}{(n-j)!\,j!}a^{n-j}x^j + \cdots \qquad [x^2 < a^2]$$

$$(a-bx)^{-1} = \frac{1}{a}\left[1 + \frac{bx}{a} + \frac{b^2x^2}{a^2} + \frac{b^3x^3}{a^3} + \cdots\right] \qquad [b^2x^2 < a^2]$$

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)x^3}{3!} + \cdots \qquad [x^2 < 1]$$

$$(1 \pm x)^{-n} = 1 \mp nx + \frac{n(n+1)}{2!}x^2 \mp \frac{n(n+1)(n+2)}{3!}x^3 + \cdots \qquad [x^2 < 1]$$

Series (continued)

$$(1 \pm x)^{1/2} = 1 \pm \frac{1}{2}x - \frac{1}{2\cdot 4}x^2 \pm \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 \pm \cdots \qquad [x^2 < 1]$$

$$(1 \pm x)^{-1/2} = 1 \mp \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 \mp \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4 \pm \cdots \qquad [x^2 < 1]$$

$$(1 \pm x^2)^{1/2} = 1 \pm \frac{1}{2}x^2 - \frac{x^4}{2\cdot 4} \pm \frac{1\cdot 3}{2\cdot 4\cdot 6}x^6 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^8 \pm \cdots \qquad [x^2 < 1]$$

$$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \cdots \qquad [x^2 < 1]$$

$$(1 \pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp \cdots \qquad [x^2 < 1]$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

$$a^x = 1 + x\log a + \frac{(x\log a)^2}{2!} + \frac{(x\log a)^3}{3!} + \cdots$$

$$\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots \qquad [0 < x < 2]$$

$$\log x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots \qquad \left[x > \frac{1}{2}\right]$$

$$\log x = 2\left[\left(\frac{x-1}{x+1}\right) + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right] \qquad [x > 0]$$

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \qquad [x^2 < 1]$$

$$\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots\right] \qquad [x^2 < 1]$$

$$\log\left(\frac{x+1}{x-1}\right) = 2\left[\frac{1}{x} + \frac{1}{3}\left(\frac{1}{x}\right)^3 + \frac{1}{5}\left(\frac{1}{x}\right)^5 + \cdots\right] \qquad [x^2 > 1]$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots + \frac{2^{2n}(2^{2n}-1)B_{2n-1}x^{2n-1}}{(2n)!} \qquad \left[x^2 < \frac{\pi^2}{4}\right]$$

$$\operatorname{ctn} x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots - \frac{B_{2n-1}(2x)^{2n}}{(2n)!\,x} - \cdots \qquad [x^2 < \pi^2]$$

Series (continued)

$$\csc x = \frac{1}{x} + \frac{x}{3!} + \frac{7x^3}{3 \cdot 5!} + \frac{31x^5}{3 \cdot 7!} + \cdots + \frac{2\left(2^{2n+1} - 1\right)}{(2n+2)!} B_{2n+1} x^{2n+1} + \cdots \qquad \left[x^2 < \pi^2\right]$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{(1 \cdot 3)x^5}{(2 \cdot 4)5} + \frac{(1 \cdot 3 \cdot 5)x^7}{(2 \cdot 4 \cdot 6)7} + \cdots \qquad \left[x^2 < 1\right]$$

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots \qquad \left[x^2 < 1\right]$$

$$\sec^{-1} x = \frac{\pi}{2} - \frac{1}{x} - \frac{1}{6x^3} - \frac{1 \cdot 3}{(2 \cdot 4)5x^5} - \frac{1 \cdot 3 \cdot 5}{(2 \cdot 4 \cdot 6)7x^7} - \cdots \qquad \left[x^2 > 1\right]$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

$$\tanh x = \left(2^2 - 1\right)2^2 B_1 \frac{x}{2!} - \left(2^4 - 1\right)2^4 B_3 \frac{x^3}{4!} + \left(2^6 - 1\right)2^6 B_5 \frac{x^5}{6!} - \cdots \qquad \left[x^2 < \frac{\pi^2}{4}\right]$$

$$\text{ctnh } x = \frac{1}{x}\left(1 + \frac{2^2 B_1 x^2}{2!} - \frac{2^4 B_3 x^4}{4!} + \frac{2^6 B_5 x^6}{6!} - \cdots\right) \qquad \left[x^2 < \pi^2\right]$$

$$\text{sech } x = 1 - \frac{B_2 x^2}{2!} + \frac{B_4 x^4}{4!} - \frac{B_6 x^6}{6!} + \cdots \qquad \left[x^2 < \frac{\pi^2}{4}\right]$$

$$\text{csch } x = \frac{1}{x} - (2-1)2B_1 \frac{x}{2!} + \left(2^3 - 1\right)2B_3 \frac{x^3}{4!} - \cdots \qquad \left[x^2 < \pi^2\right]$$

$$\sinh^{-1} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots \qquad \left[x^2 < 1\right]$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \qquad \left[x^2 < 1\right]$$

$$\text{ctnh}^{-1} x = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots \qquad \left[x^2 > 1\right]$$

$$\text{csch}^{-1} x = \frac{1}{x} - \frac{1}{2 \cdot 3x^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5x^5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7x^7} + \cdots \qquad \left[x^2 > 1\right]$$

$$\int_0^x e^{-t^2} dt = x - \frac{1}{3}x^3 + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots$$

Error Function

The following function, known as the error function, erf x, arises frequently in applications:

$$\text{erf } x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

The integral cannot be represented in terms of a finite number of elementary functions; therefore, values of erf x have been compiled in tables. The following is the series for erf x:

Series (continued)

$$\text{erf}\, x = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\right]$$

There is a close relation between this function and the area under the standard normal curve. For evaluation it is convenient to use z instead of x; then erf z may be evaluated from the area $F(z)$ by use of the relation

$$\text{erf}\, z = 2F\left(\sqrt{2}z\right)$$

Example

$$\text{erf}\,(0.5) = 2F\left[(1.414)(0.5)\right] = 2F(0.707)$$

By interpolation, $F(0.707) = 0.260$; thus, erf(0.5) = 0.520.

Series Expansion

The expression in parentheses following certain series indicates the region of convergence. If not otherwise indicated, it is understood that the series converges for all finite values of x.

Binomial

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots \qquad \left[y^2 < x^2\right]$$

$$(1\pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \frac{n(n-1)(n-2)x^3}{3!} + \cdots \qquad \left[x^2 < 1\right]$$

$$(1\pm x)^{-n} = 1 \mp nx + \frac{n(n+1)x^2}{2!} \mp \frac{n(n+1)(n+2)x^3}{3!} + \cdots \qquad \left[x^2 < 1\right]$$

$$(1\pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \cdots \qquad \left[x^2 < 1\right]$$

$$(1\pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp 6x^5 + \cdots \qquad \left[x^2 < 1\right]$$

Reversion of Series

Let a series be represented by

$$y = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots \qquad (a_1 \neq 0)$$

To find the coefficients of the series

$$x = A_1y + A_2y^2 + A_3y^3 + A_4y^4 + \cdots$$

$$A_1 = \frac{1}{a_1} \qquad A_2 = -\frac{a_2}{a_1^3} \qquad A_3 = \frac{1}{a_1^5}\left(2a_2^2 - a_1a_3\right)$$

$$A_4 = \frac{1}{a_1^7}\left(5a_1a_2a_3 - a_1^2a_4 - 5a_2^3\right)$$

$$A_5 = \frac{1}{a_1^9}\left(6a_1^2a_2a_4 + 3a_1^2a_3^2 + 14a_2^4 - a_1^3a_5 - 21a_1a_2^2a_3\right)$$

$$A_6 = \frac{1}{a_1^{11}}\left(7a_1^3a_2a_5 + 7a_1^3a_3a_4 + 84a_1a_2^3a_3 - a_1^4a_6 - 28a_1^2a_2^2a_4 - 28a_1^2a_2a_3^2 - 42a_2^5\right)$$

$$A_7 = \frac{1}{a_1^{13}}\left(8a_1^4a_2a_6 + 8a_1^4a_3a_5 + 4a_1^4a_4^2 + 120a_1^2a_2^3a_4 + 180a_1^2a_2^2a_3^2 + 132a_2^6 - a_1^5a_7 - 36a_1^3a_2^2a_5 - 72a_1^3a_2a_3a_4 - 12a_1^3a_3^3 - 330a_1a_2^4a_3\right)$$

Series (continued)

Taylor

1. $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots$ (Taylor's series)

   (Increment form)

2. $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$

   $= f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \cdots$

3. If $f(x)$ is a function possessing derivatives of all orders throughout the interval $a \le x \le b$, then there is a value X, with $a < X < b$, such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^n}{n!}f^{(n)}(X)$$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h),\ b = a+h, \quad 0<\theta<1$$

or

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + (x-a)^{n-1}\frac{f^{(n-1)}(a)}{(n-1)!} + R_n$$

where

$$R_n = \frac{f^{(n)}\left[a + \theta \cdot (x-a)\right]}{n!}(x-a)^n, \quad 0<\theta<1.$$

The above forms are known as Taylor's series with the remainder term.

4. Taylor's series for a function of two variables:

$$\text{If} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial x}\right)f(x,y) = h\frac{\partial f(x,y)}{\partial x} + k\frac{\partial f(x,y)}{\partial y};$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y) = h^2\frac{\partial^2 f(x,y)}{\partial x^2} + 2hk\frac{\partial^2 f(x,y)}{\partial x\,\partial y} + k^2\frac{\partial^2 f(x,y)}{\partial y^2}$$

etc., and if

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x,y)\bigg|_{\substack{x=a\\y=b}}$$

where the bar and subscripts mean that after differentiation we are to replace x by a and y by b,

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x,y)\bigg|_{\substack{x=a\\y=b}} + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x,y)\bigg|_{\substack{x=a\\y=b}} + \cdots$$

MacLaurin

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + x^{n-1}\frac{f^{(n-1)}(0)}{(n-1)!} + R_n$$

where

$$R_n = \frac{x^n f^{(n)}(\theta x)}{n!}, \quad 0<\theta<1$$

Series (continued)

Exponential

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad \text{(all real values of } x\text{)}$$

$$a^x = 1 + x\log_e a + \frac{(x\log_e a)^2}{2!} + \frac{(x\log_e a)^3}{3!} + \cdots$$

$$e^x = e^a\left[1 + (x-a) + \frac{(x-a)^2}{2!} + \frac{(x-a)^3}{3!} + \cdots\right]$$

Logarithmic

$$\log_e x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots \qquad \left(x > \frac{1}{2}\right)$$

$$\log_e x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots \qquad (2 \ge x > 0)$$

$$\log_e x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right] \qquad (x > 0)$$

$$\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \qquad (-1 < x \le 1)$$

$$\log_e(n+1) - \log_e(n-1) = 2\left[\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \cdots\right]$$

$$\log_e(a+x) = \log_e a + 2\left[\frac{x}{2a+x} + \frac{1}{3}\left(\frac{x}{2a+x}\right)^3 + \frac{1}{5}\left(\frac{x}{2a+x}\right)^5 + \cdots\right] \qquad (a > 0, -a < x < +\infty)$$

$$\log_e \frac{1+x}{1-x} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots\right] \qquad (-1 < x < 1)$$

$$\log_e x = \log_e a + \frac{(x-a)}{a} - \frac{(x-a)^2}{2a^2} + \frac{(x-a)^3}{3a^3} - \cdots \qquad (0 < x \le 2a)$$

Series (continued)

Trigonometric

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{(all real values of } x\text{)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{(all real values of } x\text{)}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots$$

$$+ \frac{(-1)^{n-1} 2^{2n} \left(2^{2n} - 1\right) B_{2n}}{(2n)!} x^{2n-1} + \cdots$$

$$\left(x^2 < \pi^2/4, \text{ and } B_n \text{ represents the } n\text{th Bernoulli number}\right)$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^2}{45} - \frac{2x^5}{945} - \frac{x^7}{4725} - \cdots$$

$$- \frac{(-1)^{n+1} 2^{2n}}{(2n)!} B_{2n} x^{2n-1} + \cdots$$

$$\left(x^2 < \pi^2, \text{ and } B_n \text{ represents the } n\text{th Bernoulli number}\right)$$

From Dorf, R.C., Ed., *The Engineering Handbook,* CRC Press, Boca Raton, FL, 1996, pp. 2041–2048.

Differential Calculus

Notation

For the following equations, the symbols $f(x)$, $g(x)$, etc., represent functions of x. The value of a function $f(x)$ at $x = a$ is denoted $f(a)$. For the function $y = f(x)$ the derivative of y with respect to x is denoted by one of the following:

$$\frac{dy}{dx}, \quad f'(x), \quad D_x y, \quad y'$$

Higher derivatives are as follows:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx} f'(x) = f''(x)$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx} f''(x) = f'''(x)$$

$$\vdots$$

and values of these at $x = a$ are denoted $f''(a)$, $f'''(a)$, and so on.

Slope of a Curve

The tangent line at point $P(x, y)$ of the curve $y = f(x)$ has a slope $f'(x)$ provided that $f'(x)$ exists at P. The slope at P is defined to be that of the tangent line at P. The tangent line at $P(x_1, y_1)$ is given by

$$y - y_1 = f'(x_1)(x - x_1)$$

The *normal line* to the curve at $P(x_1, y_1)$ has slope $-1/f'(x_1)$ and thus obeys the equation

$$y - y_1 = \left[-1/f'(x_1)\right](x - x_1)$$

(The slope of a vertical line is not defined.)

Angle of Intersection of Two Curves

Two curves, $y = f_1(x)$ and $y = f_2(x)$, that intersect at a point $P(X, Y)$ where derivatives $f_1'(X)$, $f_2'(X)$ exist, have an angle (α) of intersection given by

$$\tan\alpha = \frac{f_2'(X) - f_1'(X)}{1 + f_2'(X) \cdot f_1'(X)}$$

If $\tan\alpha > 0$, then α is the acute angle; if $\tan\alpha < 0$, then α is the obtuse angle.

Radius of Curvature

The radius of curvature R of the curve $y = f(x)$ at the point $P(x, y)$ is

$$R = \frac{\left\{1 + \left[f'(x)\right]^2\right\}^{3/2}}{f''(x)}$$

In polar coordinates (θ, r) the corresponding formula is

$$R = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

The *curvature* K is $1/R$.

Differential Calculus (continued)

Relative Maxima and Minima

The function f has a relative maximum at $x = a$ if $f(a) \geq f(a + c)$ for all values of c (positive or negative) that are sufficiently near zero. The function f has a relative minimum at $x = b$ if $f(b) \leq f(b + c)$ for all values of c that are sufficiently close to zero. If the function f is defined on the closed interval $x_1 \leq x \leq x_2$ and has a relative maximum or minimum at $x = a$, where $x_1 < a < x_2$, and if the derivative $f'(x)$ exists at $x = a$, then $f'(a) = 0$. It is noteworthy that a relative maximum or minimum may occur at a point where the derivative does not exist. Further, the derivative may vanish at a point that is neither a maximum nor a minimum for the function. Values of x for which $f'(x) = 0$ are called "critical values." To determine whether a critical value of x, say x_c, is a relative maximum or minimum for the function at x_c, one may use the second derivative test:

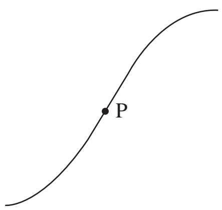
1. If $f''(x_c)$ is positive, $f(x_c)$ is a minimum.
2. If $f''(x_c)$ is negative, $f(x_c)$ is a maximum.
3. If $f''(x_c)$ is zero, no conclusion may be made.

The sign of the derivative as x advances through x_c may also be used as a test. If $f'(x)$ changes from positive to zero to negative, then a maximum occurs at x_c, whereas a change in $f'(x)$ from negative to zero to positive indicates a minimum. If $f'(x)$ does not change sign as x advances through x_c, then the point is neither a maximum nor a minimum.

Points of Inflection of a Curve

The sign of the second derivative of f indicates whether the graph of $y = f(x)$ is concave upward or concave downward:

$f''(x) > 0$: concave upward
$f''(x) < 0$: concave downward



Point of inflection.

A point of the curve at which the direction of concavity changes is called a point of inflection. Such a point may occur where $f''(x) = 0$ or where $f''(x)$ becomes infinite. More precisely, if the function $y = f(x)$ and its first derivative $y' = f'(x)$ are continuous in the interval $a \leq x \leq b$, and if $y'' = f''(x)$ exists in $a < x < b$, then the graph of $y = f(x)$ for $a < x < b$ is concave upward if $f''(x)$ is positive and concave downward if $f''(x)$ is negative.

Taylor's Formula

If f is a function that is continuous on an interval that contains a and x, and if its first $(n + 1)$ derivatives are continuous on this interval, then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R$$

where R is called the *remainder*. There are various common forms of the remainder:

Lagrange's Form

$$R = f^{(n+1)}(\beta) \cdot \frac{(x-a)^{n+1}}{(n+1)!}, \quad \beta \text{ between } a \text{ and } x$$

Cauchy's Form

$$R = f^{(n+1)}(\beta) \cdot \frac{(x-B)^n (x-a)}{n!}, \quad \beta \text{ between } a \text{ and } x$$

Differential Calculus (continued)

Integral Form

$$R = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt$$

Indeterminant Forms

If $f(x)$ and $g(x)$ are continuous in an interval that includes $x = a$, and if $f(a) = 0$ and $g(a) = 0$, the limit $\lim_{x\to a} [f(x)/g(x)]$ takes the form "0/0," called an *indeterminant form*. *L'Hôpital's rule* is

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Similarly, it may be shown that if $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

(The above holds for $x \to \infty$.)

Examples

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

Numerical Methods

1. *Newton's method* for approximating roots of the equation $f(x) = 0$: A first estimate x_1 of the root is made; then, provided that $f'(x_1) \neq 0$, a better approximation is x_2:

$$x_2 = x_1 - \frac{f(x)}{f'(x_1)}$$

The process may be repeated to yield a third approximation, x_3, to the root:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

provided $f'(x_2)$ exists. The process may be repeated. (In certain rare cases the process will not converge.)

2. *Trapezoidal rule for areas*: For the function $y = f(x)$ defined on the interval (a, b) and positive there, take n equal subintervals of width $\Delta x = (b - a)/n$. The area bounded by the curve between $x = a$ and $x = b$ [or definite integral of $f(x)$] is approximately the sum of trapezoidal areas, or

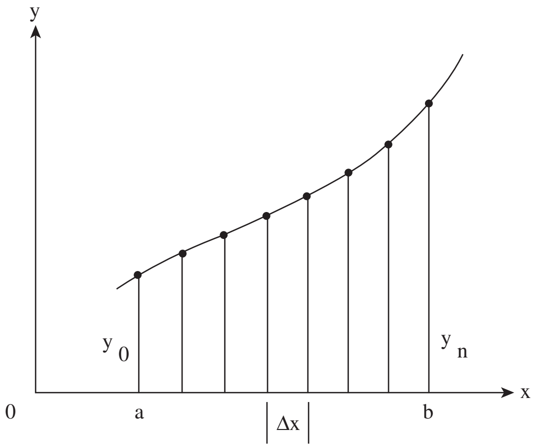
$$A \sim \left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right)(\Delta x)$$

Estimation of the error (E) is possible if the second derivative can be obtained:

$$E = \frac{b-a}{12} f''(c)(\Delta x)^2$$

where c is some number between a and b.

Differential Calculus (continued)



Trapezoidal rule for area.

Functions of Two Variables

For the function of two variables, denoted $z = f(x, y)$, if y is held constant, say at $y = y_1$, then the resulting function is a function of x only. Similarly, x may be held constant at x_1, to give the resulting function of y.

The Gas Laws

A familiar example is afforded by the ideal gas law relating the pressure p, the volume V, and the absolute temperature T of an ideal gas:

$$pV = nRT$$

where n is the number of moles and R is the gas constant per mole, 8.31 ($J \cdot K^{-1} \cdot mole^{-1}$). By rearrangement, any one of the three variables may be expressed as a function of the other two. Further, either one of these two may be held constant. If T is held constant, then we get the form known as Boyle's law:

$$p = kV^{-1} \qquad \text{(Boyle's law)}$$

where we have denoted nRT by the constant k and, of course, $V > 0$. If the pressure remains constant, we have Charles' law:

$$V = bT \qquad \text{(Charles' law)}$$

where the constant b denotes nR/p. Similarly, volume may be kept constant:

$$p = aT$$

where now the constant, denoted a, is nR/V.

Partial Derivatives

The physical example afforded by the ideal gas law permits clear interpretations of processes in which one of the variables is held constant. More generally, we may consider a function $z = f(x, y)$ defined over some region of the xy plane in which we hold one of the two coordinates, say y, constant. If the resulting function of x is differentiable at a point (x, y), we denote this derivative by one of the notations

$$f_x, \quad \delta f/dx, \quad \delta z/dx$$

called the *partial derivative with respect to x*. Similarly, if x is held constant and the resulting function of y is differentiable, we get the *partial derivative with respect to y*, denoted by one of the following:

$$f_y, \quad \delta f/dy, \quad \delta z/dy$$

Example. Given $z = x^4y^3 - y \sin x + 4y$, then

$$\delta z/dx = 4(xy)^3 - y\cos x$$

$$\delta z/dy = 3x^4y^2 - \sin x + 4$$

From Dorf, R.C., Ed., *The Engineering Handbook,* CRC Press, Boca Raton, FL, 1996, pp. 2048–2052.

Integral Calculus

Indefinite Integral

If $F(x)$ is differentiable for all values of x in the interval (a, b) and satisfies the equation $dy/dx = f(x)$, then $F(x)$ is an integral of $f(x)$ with respect to x. The notation is $F(x) = \int f(x)\,dx$ or, in differential form, $dF(x) = f(x)\,dx$.

For any function $F(x)$ that is an integral of $f(x)$, it follows that $F(x) + C$ is also an integral. We thus write

$$\int f(x)\,dx = F(x) + C$$

Definite Integral

Let $f(x)$ be defined on the interval $[a, b]$ which is partitioned by points $x_1, x_2, \ldots, x_j, \ldots, x_{n-1}$ between $a = x_0$ and $b = x_n$. The jth interval has length $\Delta x_j = x_j - x_{j-1}$, which may vary with j. The sum $\sum_{j=1}^{n} f(\upsilon_j)\Delta x_j$, where υ_j is arbitrarily chosen in the jth subinterval, depends on the numbers $x_0, \ldots, x_n$ and the choice of the v as well as f; but if such sums approach a common value as all Δx approach zero, then this value is the definite integral of f over the interval (a, b) and is denoted $\int_a^b f(x)dx$. The *fundamental theorem of integral calculus* states that

$$\int_a^b f(x)dx = F(b) - F(a),$$

where F is any continuous indefinite integral of f in the interval (a, b).

Properties

$$\int_a^b \left[f_1(x) + f_2(x) + \cdots + f_j(x)\right]dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)dx + \cdots + \int_a^b f_j(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx, \quad \text{if } c \text{ is a constant}$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Common Applications of the Definite Integral

Area (Rectangular Coordinates)

Given the function $y = f(x)$ such that $y > 0$ for all x between a and b, the area bounded by the curve $y = f(x)$, the x axis, and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b f(x)\,dx$$

Length of Arc (Rectangular Coordinates)

Given the smooth curve $f(x, y) = 0$ from point (x_1, y_1) to point (x_2, y_2), the length between these points is

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(dy/dx\right)^2}\,dx$$

$$L = \int_{y_1}^{y_2} \sqrt{1 + \left(dx/dy\right)^2}\,dy$$

Mean Value of a Function

The mean value of a function $f(x)$ continuous on $[a, b]$ is

$$\frac{1}{(b-a)}\int_a^b f(x)dx$$

Integral Calculus (continued)

Area (Polar Coordinates)

Given the curve $r = f(\theta)$, continuous and nonnegative for $\theta_1 \le \theta \le \theta_2$, the area enclosed by this curve and the radial lines $\theta = \theta_1$ and $\theta = \theta_2$ is given by

$$A = \int_{\theta_1}^{\theta_2} \tfrac{1}{2}\left[f(\theta)\right]^2 d\theta$$

Length of Arc (Polar Coordinates)

Given the curve $r = f(\theta)$ with continuous derivative $f'(\theta)$ on $\theta_1 \le \theta \le \theta_2$, the length of arc from $\theta = \theta_1$ to $\theta = \theta_2$ is

$$L = \int_{\theta_1}^{\theta_2} \sqrt{\left[f(\theta)\right]^2 + \left[f'(\theta)\right]^2}\, d\theta$$

Volume of Revolution

Given a function $y = f(x)$ continuous and nonnegative on the interval (a, b), when the region bounded by $f(x)$ between a and b is revolved about the x axis, the volume of revolution is

$$V = \pi \int_a^b \left[f(x)\right]^2 dx$$

***Surface Area of Revolution* (Revolution about the *x* Axis, Between *a* and *b*)**

If the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ is revolved about the x axis, the area A of the surface generated is given by the following:

$$A = \int_a^b 2\pi f(x)\left\{1 + \left[f'(x)\right]^2\right\}^{1/2} dx$$

Work

If a variable force $f(x)$ is applied to an object in the direction of motion along the x axis between $x = a$ and $x = b$, the work done is

$$W = \int_a^b f(x)\, dx$$

Cylindrical and Spherical Coordinates

1. Cylindrical coordinates:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

   Element of volume $dV = r\, dr\, d\theta\, dz$.

2. Spherical coordinates:

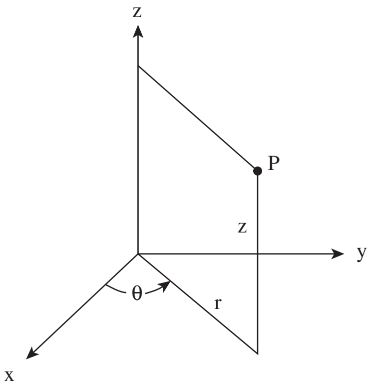
$$x = \rho\sin\phi\cos\theta$$

$$y = \rho\sin\phi\sin\theta$$

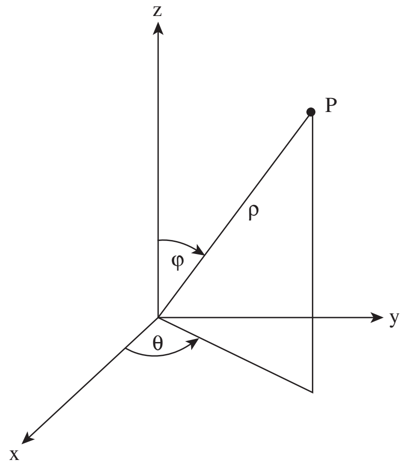
$$z = \rho\cos\phi$$

   Element of volume $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

Integral Calculus (continued)



Cylindrical coordinates.



Spherical coordinates.

Double Integration

The evaluation of a double integral of $f(x, y)$ over a plane region R,

$$\iint_R f(x,y)dA$$

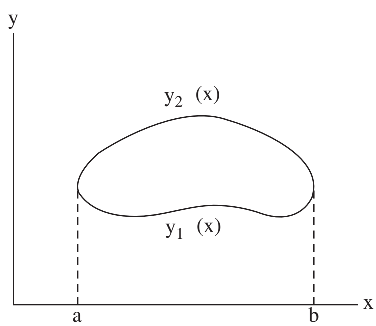
is practically accomplished by iterated (repeated) integration. For example, suppose that a vertical straight line meets the boundary of R in at most two points so that there is an upper boundary, $y = y_2(x)$, and a lower boundary, $y = y_1(x)$. Also, it is assumed that these functions are continuous from a to b (see figure below). Then

$$\iint_R f(x,y)dA = \int_a^b \left(\int_{y_1(x)}^{y_2(y)} f(x,y)dy \right) dx$$

If R has left-hand boundary, $x = x_1(y)$, and right-hand boundary, $x = x_2(y)$, which are continuous from c to d (the extreme values of y in R), then

$$\iint_R f(x,y)dA = \int_c^d \left(\int_{x_1(y)}^{x_2(y)} f(x,y)dx \right) dy$$

Such integrations are sometimes more convenient in polar coordinates, $x = r \cos \theta$, $y = r \sin \theta$, $dA = r\, dr\, d\theta$.



Region R bounded by $y_2(x)$ and $y_1(x)$.

Integral Calculus (continued)

Surface Area and Volume by Double Integration

For the surface given by $z = f(x, y)$, which projects onto the closed region R of the xy plane, one may calculate the volume V bounded above by the surface and below by R, and the surface area S by the following:

$$V = \iint_R z\,dA = \iint_R f(x,y)\,dx\,dy$$

$$S = \iint_R \left[1 + (\delta z/\delta x)^2 + (\delta z/\delta y)^2\right]^{1/2} dx\,dy$$

[In polar coordinates, (r, θ), we replace $d\,A$ by $r\,d\,r\,d\,\theta$.]

Centroid

The centroid of a region R of the xy plane is a point (x', y') where

$$x' = \frac{1}{A}\iint_R x\,d\,A, \qquad y' = \frac{1}{A}\iint_R y\,d\,A$$

and A is the area of the region.

Example. For the circular sector of angle 2α and radius R, the area A is αR^2; the integral needed for x', expressed in polar coordinates, is

$$\iint x\,dA = \int_{-\alpha}^{\alpha}\int_0^R (r\cos\theta)\,r\,dr\,d\theta$$

$$= \left[\frac{R^3}{3}\sin\theta\right]_{-\alpha}^{+\alpha} = \frac{2}{3}R^3\sin\theta$$

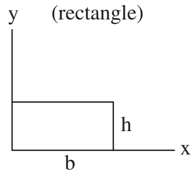
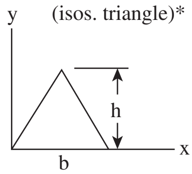
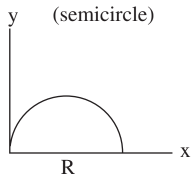
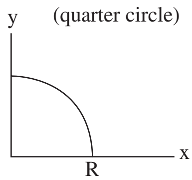
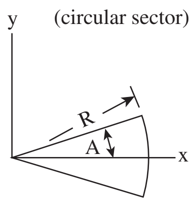
and thus,

$$x' = \frac{\frac{2}{3}R^3\sin\alpha}{\alpha R^2} = \frac{2}{3}R\frac{\sin\alpha}{\alpha}$$

Centroids of some common regions are shown in the following table.

Integral Calculus (continued)

Centroids

	Area	x'	y'
Rectangle	bh	$b/2$	$h/2$
			
Isosceles triangle*	$bh/2$	b/2	h/3
			
Semicircle	$\pi R^2/2$	R	$4R/3\pi$
			
Quarter circle	$\pi R^2/4$	4R/3π	$4R/3\pi$
			
Circular sector	R^2A	$2R \sin A/3A$	0
			

* $y' = h/3$ for any triangle of altitude h.

From Dorf, R.C., Ed., *The Engineering Handbook,* CRC Press, Boca Raton, FL, 1996, pp. 2053–2057.

Special Functions

Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \operatorname{ctnh} x = \frac{1}{\tanh x}$$

$$\sinh(-x) = -\sinh x \qquad \operatorname{ctnh}(-x) = -\operatorname{ctnh} x$$

$$\cosh(-x) = \cosh x \qquad \operatorname{sech}(-x) = \operatorname{sech} x$$

$$\tanh(-x) = -\tanh x \qquad \operatorname{csch}(-x) = -\operatorname{csch} x$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \operatorname{ctnh} x = \frac{\cosh x}{\sinh x}$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad \cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1) \qquad \operatorname{ctnh}^2 x - \operatorname{csch}^2 x = 1$$

$$\operatorname{csch}^2 x - \operatorname{sech}^2 x = \operatorname{csch}^2 x \operatorname{sech}^2 x \qquad \tanh^2 x + \operatorname{sech}^2 x = 1$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$$

$$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$$

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

$$\tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$$

Bessel Functions

Bessel functions, also called cylindrical functions, arise in many physical problems as solutions of the differential equation

$$x^2 y'' + xy' + \left(x^2 - n^2\right) y = 0$$

which is known as Bessel's equation. Certain solutions, known as *Bessel functions of the first kind of order n*, are given by

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{-n+2k}$$

In the above it is noteworthy that the gamma function must be defined for the negative argument $q : \Gamma(q) = \Gamma(q+1)/q$, provided that q is not a negative integer. When q is a negative integer, $1/\Gamma(q)$ is defined to be zero. The functions $J_{-n}(x)$ and $J_n(x)$ are solutions of Bessel's equation for all real n. It is seen, for $n = 1, 2, 3, \ldots$, that

$$J_{-n}(x) = (-1)^n J_n(x)$$

and, therefore, these are not independent; hence, a linear combination of these is not a general solution. When, however, n is not a positive integer, a negative integer, or zero, the linear combination with arbitrary constants c_1 and c_2,

Special Functions (continued)

$$y = c_1 J_n(x) + c_2 J_{-n}(x)$$

is the general solution of the Bessel differential equation.

The zero-order function is especially important as it arises in the solution of the heat equation (for a "long" cylinder):

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots$$

while the following relations show a connection to the trigonometric functions:

$$J_{1/2}(x) = \left[\frac{2}{\pi x}\right]^{1/2} \sin x$$

$$J_{-1/2}(x) = \left[\frac{2}{\pi x}\right]^{1/2} \cos x$$

The following recursion formula gives $J_{n+1}(x)$ for any order in terms of lower-order functions:

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

Legendre Polynomials

If Laplace's equation, $\nabla^2 V = 0$, is expressed in spherical coordinates, it is

$$r^2 \sin\theta \frac{\delta^2 V}{\delta r^2} + 2r \sin\theta \frac{\delta V}{\delta r} + \sin\theta \frac{\delta^2 V}{\delta \theta^2} + \cos\theta \frac{\delta V}{\delta \theta} + \frac{1}{\sin\theta} \frac{\delta^2 V}{\delta \phi^2} = 0$$

and any of its solutions, $V(r, \theta, \phi)$, are known as *spherical harmonics*. The solution as a product

$$V(r, \theta, \phi) = R(r)\Theta(\theta)$$

which is independent of ϕ, leads to

$$\sin^2\theta\, \Theta'' + \sin\theta \cos\theta\, \Theta' + \left[n(n+1)\sin^2\theta\right]\Theta = 0$$

Rearrangement and substitution of $x = \cos\theta$ leads to

$$\left(1 - x^2\right)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + n(n+1)\Theta = 0$$

known as *Legendre's equation*. Important special cases are those in which n is zero or a positive integer, and, for such cases, Legendre's equation is satisfied by polynomials called Legendre polynomials, $P_n(x)$. A short list of Legendre polynomials, expressed in terms of x and $\cos\theta$, is given below. These are given by the following general formula:

$$P_n(x) = \sum_{j=0}^{L} \frac{(-1)^j (2n - 2j)!}{2^n j!(n-j)!(n-2j)!} x^{n-2j}$$

where $L = n/2$ if n is even and $L = (n - 1)/2$ if n is odd.

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

Special Functions (continued)

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$

$$P_0(\cos\theta) = 1a$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \frac{1}{4}(3\cos 2\,\theta + 1)$$

$$P_3(\cos\theta) = \frac{1}{8}(5\cos 3\,\theta + 3\cos\theta)$$

$$P_4(\cos\theta) = \frac{1}{64}(35\cos 4\,\theta + 20\cos 2\theta + 9)$$

Additional Legendre polynomials may be determined from the *recursion formula*

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \quad (n = 1, 2, \ldots)$$

or the *Rodrigues formula*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2 - 1\right)^n$$

Laguerre Polynomials

Laguerre polynomials, denoted $L_n(x)$, are solutions of the differential equation

$$xy'' + (1 - x)y' + ny = 0$$

and are given by

$$L_n(x) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} C_{(n,j)} x^j (n = 0, 1, 2, \ldots)$$

Thus,

$$L_0(x) = 1$$

$$L_1(x) = 1 - x$$

$$L_2(x) = 1 - 2x + \frac{1}{2}x^2$$

$$L_3(x) = 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3$$

Additional Laguerre polynomials may be obtained from the recursion formula

$$(n+1)L_{n+1}(x) - (2n + 1 - x)L_n(x) + nL_{n-1}(x) = 0$$

Hermite Polynomials

The Hermite polynomials, denoted $H_n(x)$, are given by

$$H_0 = 1, \qquad H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}, \qquad (n = 1, 2, \ldots)$$

and are solutions of the differential equation

$$y'' - 2xy' + 2ny = 0 \quad (n = 0, 1, 2\ldots)$$

Special Functions (continued)

The first few Hermite polynomials are

$$H_0 = 1 \qquad H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2 \qquad H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

Additional Hermite polynomials may be obtained from the relation

$$H_{n+1}(x) = 2xH_n(x) - H_n'(x)$$

where prime denotes differentiation with respect to x.

Orthogonality

A set of functions $\{f_n(x)\}$ ($n = 1, 2, \ldots$) is orthogonal in an interval (a, b) with respect to a given weight function $w(x)$ if

$$\int_a^b w(x) f_m(x) f_n(x) dx = 0 \quad \text{when } m \neq n$$

The following polynomials are orthogonal on the given interval for the given $w(x)$:

Legendre polynomials:	$P_n(x)$	$w(x) = 1$
		$a = -1, b = 1$
Laguerre polynomials:	$L_n(x)$	$w(x) = \exp(-x)$
		$a = 0, b = \infty$
Hermite polynomials:	$H_n(x)$	$w(x) = \exp(-x^2)$
		$a = -\infty, b = \infty$

The Bessel functions *of order n*, $J_n(\lambda_1 x)$, $J_n(\lambda_2 x)$, . . . , are orthogonal with respect to $w(x) = x$ over the interval (0, c) provided that the λ_i are the positive roots of $J_n(\lambda c) = 0$:

$$\int_0^c x J_n(\lambda_j x) J_n(\lambda_k x) dx = 0 \quad (j \neq k)$$

where n is fixed and $n \geq 0$.

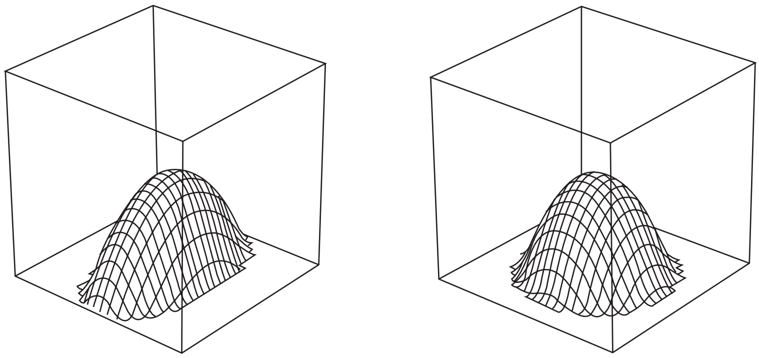
Special Functions (continued)

Functions with $x^2/a^2 \pm y^2/b^2$

Elliptic Paraboloid

$$z = c\left(x^2/a^2 + y^2/b^2\right)$$

$$x^2/a^2 + y^2/b^2 - z/c = 0$$



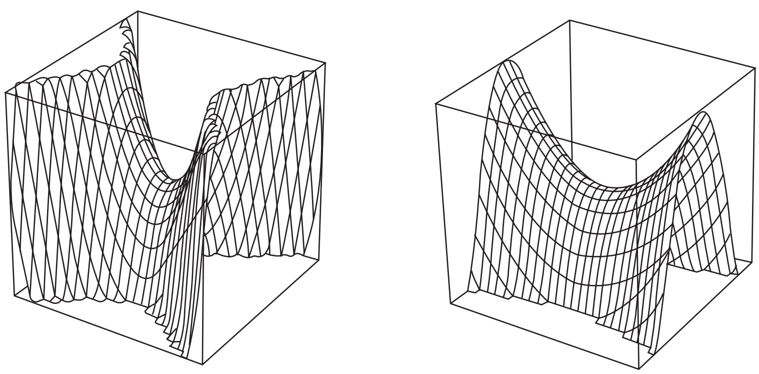
(a) (b)

Elliptic paraboloid. (a) $a = 0.5$, $b = 1.0$, $c = -1.0$; viewpoint = (5, –6, 4). (b) $a = 1.0$, $b = 1.0$, $c = -2.0$; viewpoint = (5, –6, 4).

Hyperbolic Paraboloid (Commonly Called Saddle)

$$z = c\left(x^2/a^2 - y^2/b^2\right)$$

$$x^2/a^2 - y^2/b^2 - z/c = 0$$



(a) (b)

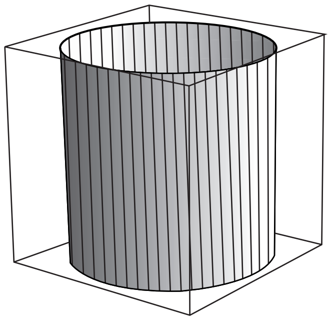
Hyperbolic paraboloid. (a) $a = 0.50$, $b = 0.5$, $c = 1.0$; viewpoint = (4, –6, 4). (b) $a = 1.00$, $b = 0.5$, $c = 1.0$; viewpoint = (4, –6, 4).

Special Functions (continued)

Elliptic Cylinder

$$1 = x^2/a^2 + y^2/b^2$$

$$x^2/a^2 + y^2/b^2 - 1 = 0$$

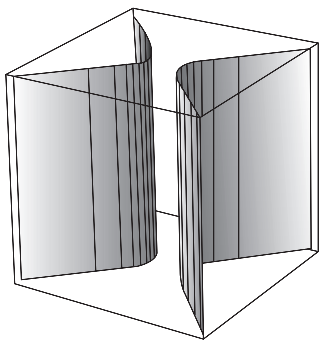


Elliptic cylinder. a = 1.0, b = 1.0; viewpoint = (4, –5, 2).

Hyperbolic Cylinder

$$1 = x^2/a^2 - y^2/b^2$$

$$x^2/a^2 - y^2/b^2 - 1 = 0$$



Hyperbolic cylinder. a = 1.0, b = 1.0; viewpoint = (4, –6, 3).

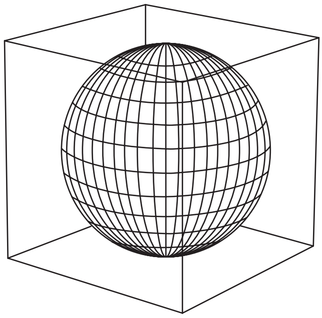
Special Functions (continued)

Functions with $(x^2/a^2 + y^2/b^2 \pm c^2)^{1/2}$

Sphere

$$z = \left(1 - x^2 - y^2\right)^{1/2}$$

$$x^2 + y^2 + z^2 - 1 = 0$$

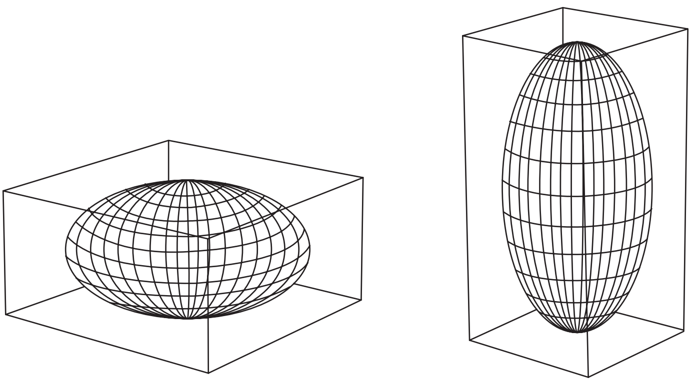


Sphere. Viewpoint = (4, –5, 2).

Ellipsoid

$$z = c\left(1 - x^2/a^2 - y^2/b^2\right)^{1/2}$$

$$x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 = 0$$



(a) (b)

Ellipsoid. (a) a = 1.00, b = 1.00, c = 0.5; viewpoint = (4, –5, 2). (b) a = 0.50, b = 0.50, c = 1.0; viewpoint = (4, –5, 2).

Special cases:

$a = b > c$ gives oblate spheroid

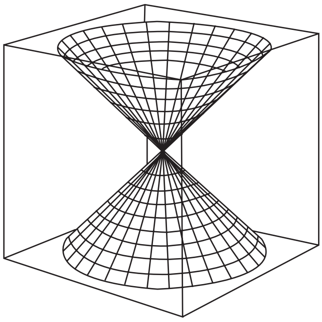
$a = b < c$ gives prolate spheroid

Special Functions (continued)

Cone

$$z = \left(x^2 + y^2\right)^{1/2}$$

$$x^2 + y^2 - z^2 = 0$$

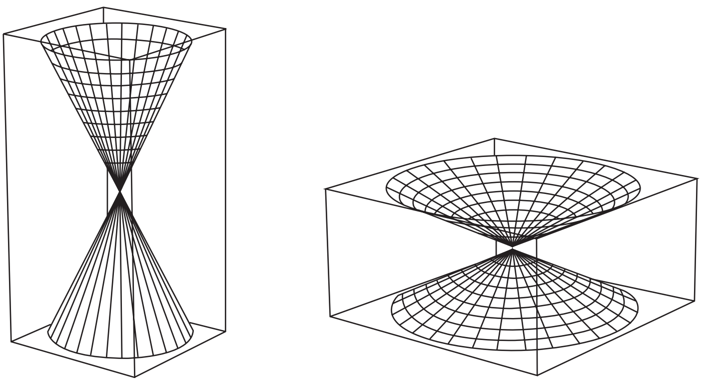


Cone. Viewpoint = (4, –5, 2).

Elliptic Cone (Circular Cone if a = b)

$$z = c\left(x^2/a^2 + y^2/b^2\right)^{1/2}$$

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 0$$



(a) (b)

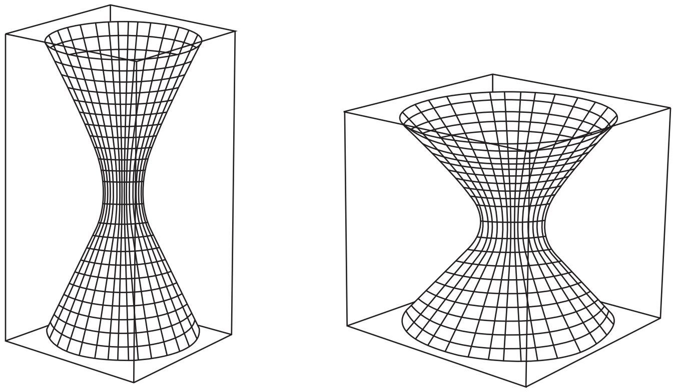
Elliptic cone. (a) a = 0.5, b = 0.5, c = 1.00; viewpoint = (4, –5, 2). (b) a = 1.0, b = 1.0, c = 0.50; viewpoint = (4, –5, 2).

Special Functions (continued)

Hyperboloid of One Sheet

$$z = c\left(x^2/a^2 + y^2/b^2 - 1\right)^{1/2}$$

$$x^2/a^2 + y^2/b^2 - z^2/c^2 - 1 = 0$$



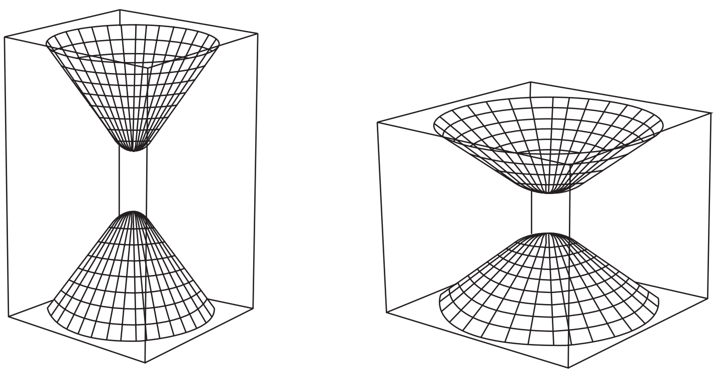
(a) (b)

Hyperboloid of one sheet. (a) $a = 0.1$, $b = 0.1$, $c = 0.2$; $\pm z = c\sqrt{15}$; viewpoint = (4, –5, 2). (b) $a = 0.2$, $b = 0.2$, $c = 0.2$; $\pm z = c\sqrt{15}$; viewpoint = (4, –5, 2).

Hyperboloid of Two Sheets

$$z = c\left(x^2/a^2 + y^2/b^2 + 1\right)^{1/2}$$

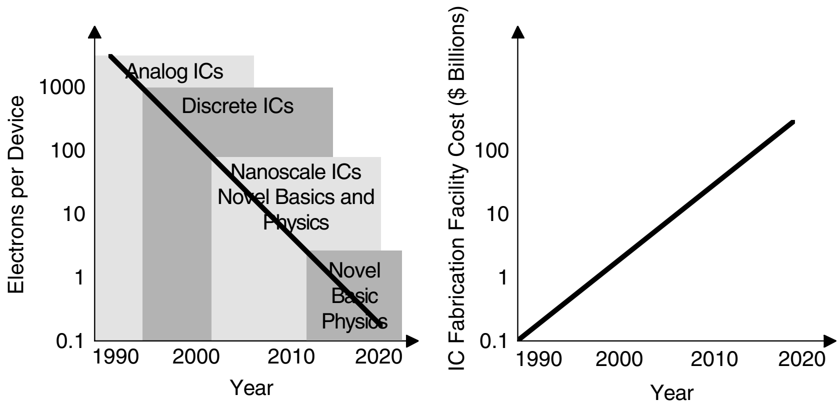
$$x^2/a^2 + y^2/b^2 - z^2/c^2 + 1 = 0$$



(a) (b)

Hyperboloid of two sheets. (a) $a = 0.125$, $b = 0.125$, $c = 0.2$; $\pm z = c\sqrt{17}$; viewpoint = (4, –5, 2). (b) $a = 0.25$, $b = 0.25$, $c = 0.2$; $\pm z = c\sqrt{17}$; viewpoint = (4, –5, 2).

From Dorf, R.C., Ed., *The Engineering Handbook,* CRC Press, Boca Raton, FL, 1996, pp. 2058–2066.



Moore's laws. (From Lyshevski, S.E., Nanocomputer architectures and nanotechnology, in *Handbook of Nanoscience, Engineering, and Technology,* Goddard, III, W.A., Brenner, D.W., Lyshevski, S.E., and Iafrate, G.J., Eds., CRC Press, Boca Raton, FL, 2003, p. **6**-5.)

Approximate Current Densities in Electrons per Second per Square Nanometer Calculated from Experimental Data for Selected Molecular Electronic and Macroscopic Metal Devices

Quantity	Units	Molecular Electronic Device: 1,4-Dithiol Benzene	3-Ring Poly-phenylene Wire	Poly-phenylene RTD (5 rings)	Carbon Nanotube	Copper Wire
Applied Voltage	Volts	1	1	1.4 (peak)	1	2×10^{-3} (10 cm wire)
Current Measured in Experiment	Amperes	2×10^{-8}	3.2×10^{-5}	1.4×10^{-11}	1×10^{-7}	1 (approx.)
Current Inferred	Amperes	2×10^{-8}	3.2×10^{-8}	1.4×10^{-14}	1×10^{-7}	–
per Molecule	Electrons per Sec	1.2×10^{11}	2.0×10^{11}	8.7×10^{4}	6.2×10^{11}	–
Estimated Cross-Sectional Area per Molecule	nm^2	~0.05	~0.05	~0.05	~3.1 (Radius ≈ 1 nm)	~3.1×10^{12} (Radius ≈ 1 mm)
Current Density	Electrons per Sec-nm^2	~2×10^{12}	~4×10^{12}	~2×10^{6}	~2×10^{11}	~2×10^{6}
Reference		(7)	(8)	(5,6)	(4)	

[a] Conversion factor for amperes to electrons per second is 1 Ampere ≡ 1 Coulomb/sec = $(1.6 \times 10^{-19})^{-1}$ electrons/sec = 6.2×10^{18} electrons/sec.

[b] In order to estimate the current densities per molecule from the published data on the room temperature nanopore measurements in References 5, 6, and 8, it was determined that the samples in the monolayer in the nanopore contained on the order of 1000 molecules per monolayer. This estimate is based on an average nanopore diameter of 30 nm and an estimated molecular diameter on the order of approximately 1 nm.

[c] Common copper wire generally is regarded as being highly conductive. Therefore, data for 10 cm of 1mm diameter (18 gauge) copper wire is included only for comparison as a familiar, conductive, macroscopic reference system. A current on the order of 1 ampere is the maximum recommended for such wire to avoid undue heating and danger of fire.

Sources of current measurements: (4) S.J. Tans et al., Individual single-wall carbon nanotubes as quantum wires, *Nature*, 386, 474–477, 1997; (5) M.A. Reed, Electrical Properties of Molecular Devices, presented at 1997 DARPA ULTRA Program Review Conference, Santa Fe, NM, October, 1997; (6) M.A. Reed, Molecular-scale electronics, *Proc. IEEE*, 87, 652–658, 1999; (7) C. Thou, M.R. Deshpande, M.A. Reed, and J.M. Tour, Nanoscale metal/self-assembled monolayer/metal heterostructures, *Appl. Phys. Lett.*, 71, 611–613, 1997; (8) C. Zhou, Atomic and Molecular Wires, Ph.D. dissertation, Yale University, 1999. With permission.

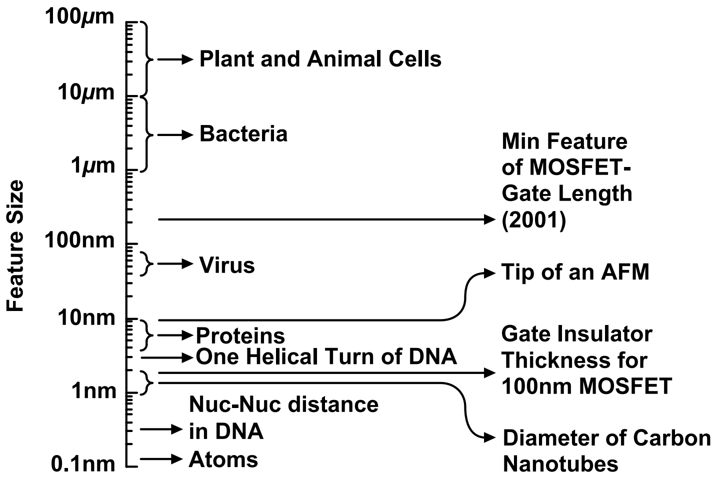
From Ellenbogen, J.C. and Love, J.C., Architectures for molecular electronic computers, in *Handbook of Nanoscience, Engineering, and Technology,* Goodard, III, W.A., Brenner, D.W., Lyshevski, S.E., and Iafrate, G.J., Eds., CRC Press, Boca Raton, FL, 2003, p. **7**-6.

Comparison of Memory Technologies for the Year 2011

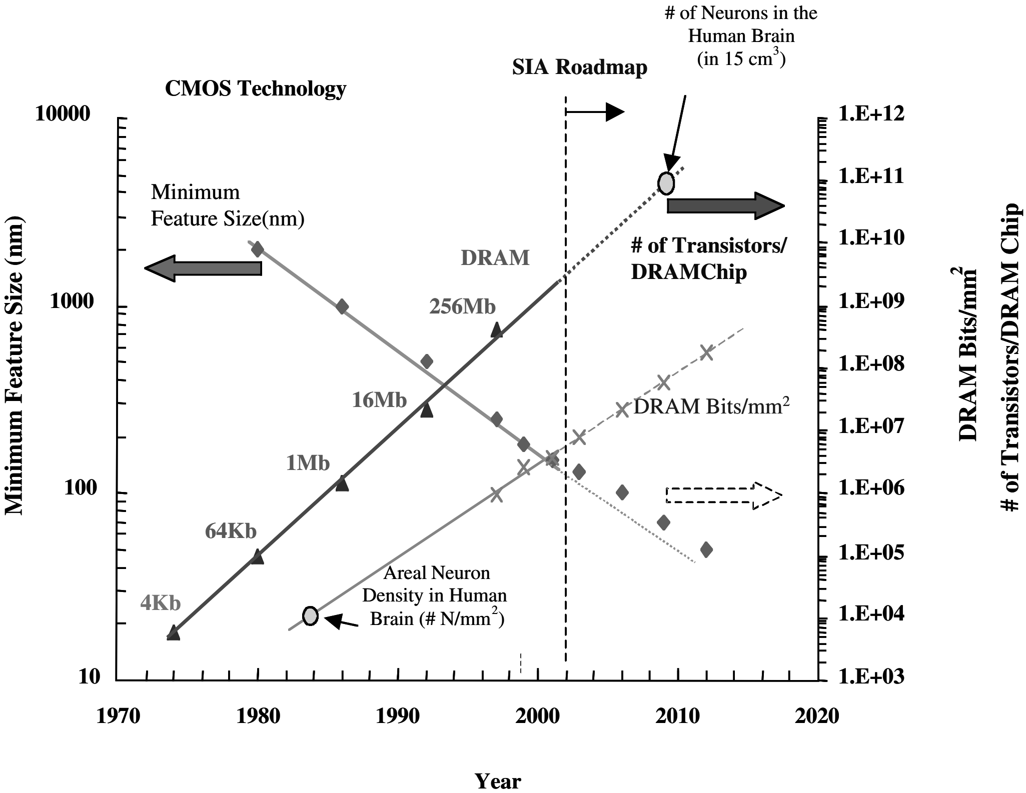
	CMOS			
Technology	DRAM	Flash	SRAM	MRAM
Reference	SIA 1999	SIA 1999	SIA 1999	
Generation at Introduction	64 GB	64 GB	180 MB/cm^2	64 GB
Circuit Speed	150 MHz	150 MHz	913 MHz	>500 MHz
Feature Size	50 nm	50 nm	35 nm	<50 nm
Access Time	10ns	10 ns	1.1 ns	<2 ns
Write Time	10 ns	10 µs	1.1 ns	<10 ns
Erase Time	<1 ns	10 µs	1.1 ns	N/A
Retention Time	2–4 s	10 years	N/A	Infinite
Endurance Cycles	Infinite	10^5	Infinite	Infinite
Operating Voltage (V)	0.5–0.6 V	5 V	0.5–0.6 V	<1 V
Voltage to Switch State	0.2 V	5 V	0.5–0.6 V	<50 mV
Cell Size	2.5 F^2*/bit 0.0005 µm^2	2F^2/bit	12F^2/bit	2F^2/bit

* F = minimal lithographic feature size.

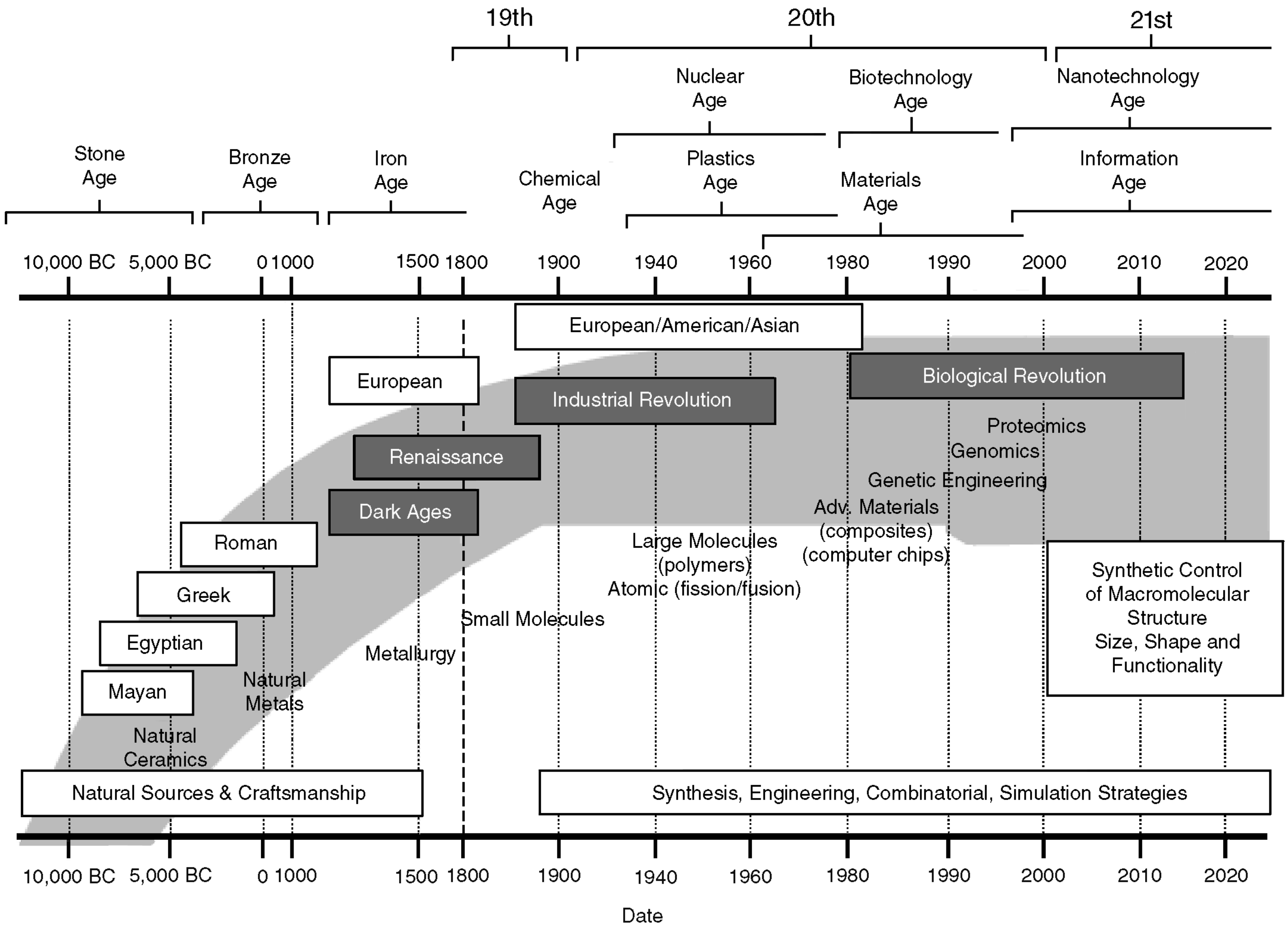
From Wolf, S.A., Chtchelkanova, A.Y., and Treger, D., Spintronics — Spin-based electronics, in *Handbook of Nanoscience, Engineering, and Technology,* Goodard, III, W.A., Brenner, D.W., Lyshevski, S.E., and Iafrate, G.J., Eds., CRC Press, Boca Raton, FL, 2003, p. **8**-6.



Size and scale of naturally occurring structures as compared with human-made structures. (From Bashir, R., Biologically mediated assembly of artificial nanostructures and microstructures, in *Handbook of Nanoscience, Engineering, and Technology,* Goddard, III, W.A., Brenner, D.W., Lyshevski, S.E., and Iafrate, G.J., Eds., CRC Press, Boca Raton, FL, 2003, p. **15**-2.)



Trends in miniaturization of integrated circuits in the last 25 years. (From Bashir, R., Biologically mediated assembly of artificial nanostructures and microstructures, in *Handbook of Nanoscience, Engineering, and Technology,* Goddard, III, W.A., Brenner, D.W., Lyshevski, S.E., and Iafrate, G.J., Eds., CRC Press, Boca Raton, FL, 2003, p. **15**-3.)



Civilizations, technology periods (ages), and historical revolutions as a function of time. (From Tomalia, D.A., Mardel, K., Henderson, S.A., Holan, G., and Estard, R., Dendrimers — An enabling synthetic science to controlled organic nanostructures, in *Handbook of Nanoscience, Engineering, and Technology,* Goddard, III, W.A., Brenner, D.W., Lyshevski, S.E., and Iafrate, G.J., Eds., CRC Press, Boca Raton, FL, 2003, p. **20**-3.)

Abbreviations

AAAS	American Association for Advancement of Science
ACS	American Chemical Society
AGI	American Geological Institute
AIChE	American Institute of Chemical Engineers
AIME	American Institute of Mining, Metallurgical and Petroleum Engineers
AIP	American Institute of Physics
AMA	actual mechanical advantage
API	American Petroleum Institute
ASCE	American Society of Civil Engineers
ASME	American Society of Mechanical Engineers
BET	Brunauer-Emmet-Teller
DOD	Department of Defense (U.S.)
DOE	Department of Energy (U.S.)
e.g.	for example
eng., engr.	engineer, engineering
erf	error function
esp.	especially
est.	estimate(d)
etc.	and so forth
exp(x)	(x) is the exponent of e
ff.	and following
i.e.	that is
ISO	International Standards Organization
LH	latent heat
ln	logarithm to the base e
log	logarithm to the base 10
MTBF	mean time before failure (same as)
MTTF	mean time to failure
n.a.	not available
n.d.	no date; or undated
NAE	National Academy of Engineering (U.S.)
NAS	National Academy of Sciences (U.S.)
NASA	National Aeronautics and Space Administration (U.S.)
NBS	National Bureau of Standards (presently NIST)
NIST	National Institute of Standards and Technology (U.S.)
NIH	National Institutes of Health (U.S.)
NSF	National Science Foundation (U.S.)
NTP	normal temperature and pressure (25°C at 1 atm.)
NUC	National Union Catalogue
NYPL	New York Public Library
TMA	theoretical mechanical advantage

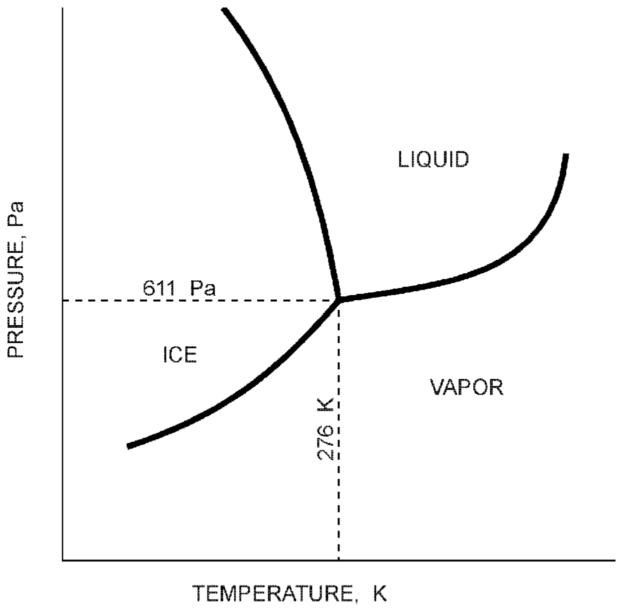
From Hall, C.W., *Laws and Models: Science, Engineering, and Technology,* CRC Press, Boca Raton, FL, 2000, p. xxvii.

Boiling Point Law, General

The decrease in vapor pressure of a nonvolatile solvent at the boiling point is proportional to the increase in mole fraction of solute as related to the moles of solute present. Thus:

$$-dp = -p\ dx/x$$

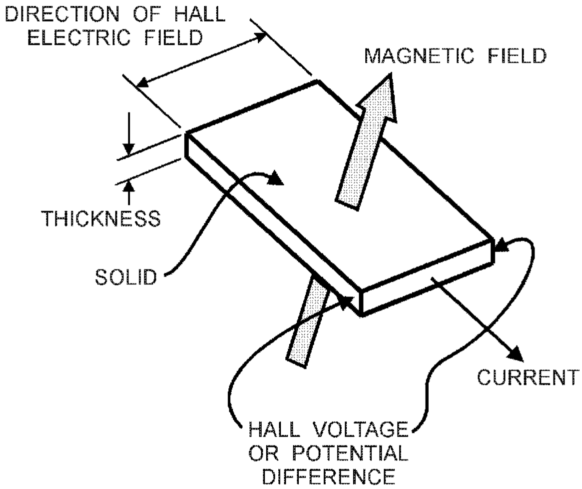
where p = vapor pressure
x = mole of solute



Triple point for water.

Hall Effect (1879)

When a steady current is flowing in a steady magnetic field, electromotive forces (voltages) are developed at right angles both to the magnetic force and to the current, and these are proportional to the product of the intensity of the current, the magnetic force, and the sine of the angle between the directions of these quantities.



Ideal Mixtures, Law of

The property of a mixture of gases and some solids and some liquids is an additive function of the same property of the components, an assumption which in many cases is far from correct:

$$W\,A = W_1\,A_1 + W_2\,A_2 + W_3\,A_3 + \ldots$$

Large Numbers, Law of (1689) (1713); Bernoulli Theorem (1713)

Various statements are used to represent the law of large numbers, but the idea is the same for each. If the size of a sample of statistically independent variables is increased indefinitely, good sample estimates of population parameters will tend to concentrate more and more closely about the true value. There are strong laws and weak laws of large numbers. Strong laws are concerned with showing that a variable, x, converges to a value μ with a probability of one. The strong law of large numbers is represented by the Borel theorem.

Weak laws consider conditions under which the probability that $|x - \mu|$ is greater than some given epsilon, ε, tends to zero. The weak law of large numbers is represented by the Bernoulli theorem. For the Bernoulli theorem, we have the following relationship:

$$\lim P\,(|x - m| > \varepsilon) = 0$$

$$\text{as } N \to \infty$$

where $\bar{x}$ = sample means
m = population
N = number of trials

S. Poisson gave the name of the law of large numbers to J. Bernoulli.

Maxwell Electromagnetic Field Equations (1864)

Maxwell, who was born in 1831, the year that Faraday published his discovery of electromagnetic induction, expressed Faraday's discoveries mathematically in field theory in 1855–1857, followed by his major work in four electromagnetic field equations, expressed in vector form as follows:

1. $\nabla \cdot D = \rho$, where D is electric displacement. The electric flux lines, if they end, will end on electric charges.
2. $\nabla \cdot \beta = 0$, where β is magnetic flux density. Those magnetic flux lines never terminate.
3. $\nabla \cdot E = -\delta\beta/\delta t$, which is a form of the Faraday law of induction, where E is the electric field density.
4. $\nabla \cdot H = i + \delta D/\delta t$, where H is magnetic field density. Based on the work by Ampere on steady currents, it shows that the line integral of magnetic intensity around a closed curve equals the current encircled, i.

Moore Law (1964)

Integrated circuits and microelectronics will double in density every other year (or every one and half years, by some references), according to a binary growth curve, and the design cost and number of functions per circuit will keep pace with complexity (on a 1960 to 1985 time frame). Elements per chip = $2^{(\text{years}/\tau)}$ where τ is 1, 2, 3, … years.

Newton Laws of Motion (Three Laws) (1687); Laws of Dynamics

These three basic laws form the basis of classical mechanics; that is, for mechanical problems not involving atomic particles or smaller, and speeds not involving the speed of light. The first law is a restatement of the discovery by Galileo that no force is required for steady, unchanging motion.

First Law (Law of Inertia)
A body at rest remains at rest, a body in motion continues in motion at constant speed along a straight line, unless the body, whether at rest or moving, is acted upon by an unbalanced force.

Second Law (Law of Constant Acceleration)
An unbalanced force acting on a body causes the body to accelerate in the direction of the force, with the acceleration directly proportional to the mass (m) of the body:

$$F = m\,a = W/g\,a$$

where F = unbalanced force
m = mass of the body
W = weight of the body
a = acceleration
g = gravitational constant

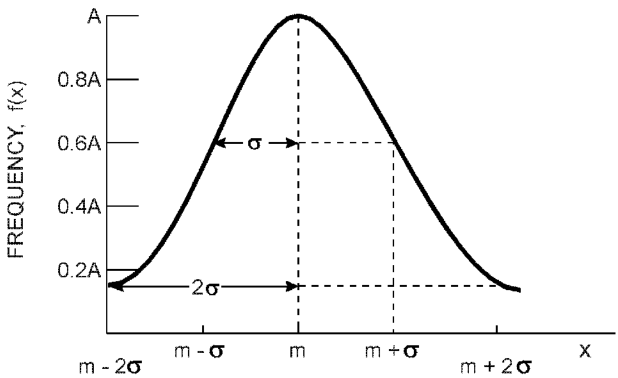
Third Law [Law of Conservation of Momentum (Motion)]
For every action there is an equal and opposite reaction, applies to all forces—electrical, gravitational, magnetic, etc.

Normal Law; Normal Law of Error; Normal or Gaussian Distribution Law; Gauss Error Curve; Probability Curve

The Gaussian distribution and the normal law of error are both often expressed as the same relationship. The Gaussian distribution law is the theoretical frequency distribution for a set of data of any normal, repetitive function, due to chance, represented by a bell-shaped curve symmetrical about a mean. The relationship of the number of events occuring and frequency when the events occur are due to chance only. The probability for distributions that occur due to chance is:

$$f(x) = p = h/(\pi)^{1/2} \exp(-h^2x^2)$$

where p = the probability, often written as y = p
h = a constant that depends on spread of the data or is a measure of precision
x = distance, plus or minus, from the center



Gaussian distribution and normal distribution.

Photoelectric Effect, Laws of (1888)(1924); (Published in 1930)

1. For a given frequency of incident light, the kinetic energy of ejected photoelectrons does not change, but their number increases in direct proportion to the light intensity.
2. When the frequency of incident light changes (increases), no electrons are emitted until a certain threshold frequency is reached (depending on the metal). For higher frequencies, the energy of photoelectrons increases in direct proportion to the difference of the frequency used and the threshold frequencies. This is represented by the Einstein equation:

$$W = h\,\gamma - \phi$$

where W = maximum kinetic energy given off by electrons
h = Planck constant
γ = frequency
ϕ = minimum energy to remove an electron from a solid (can also be applied to a gas)

Shannon Law or Formula or Theorem (1948)

The information transmitted from a message source over a communications system is represented by:

$$C = W \log_2(1 + P/N)$$

where C = channel capacity in bits per second
W = bandwidth
P = signal power
N = gaussian noise power, N = kTW where k = Boltzmann constant and T = temperature

Skin Effect

In any current carrying conductor the current tends to concentrate toward the outer surface as a result of eddy currents. An alternating current flowing through a pipe produces a magnetic field which produces eddy currents. Eddy current losses, the heat generated by eddy currents is given by:

$$P_e = K_e f^2 \beta_m^2 V_{ol}$$

where P_e = power loss by eddy current
K_e = eddy current loss constant
f = frequency of electricity
β_m = maximum flux density (webers/m^2)
V_{ol} = volume of pipe (conductor)

Snell Law (1613*, 1621); Snell Law of Refraction; Descartes Law

The relationship of the angle of incidence, the angle of refraction, the velocity light in a first medium, and the velocity in the second medium, gives the index of refraction:

$$n = \sin i/\sin r \qquad i = v/v'$$

where n = index of refraction
i = angle of incidence
r = angle of refraction
v = velocity in first medium
v′ = velocity in second medium

Although the relationship was discovered in 1621, as stated above, and one author claims 1613*, the phenomenon was known at the time of Ptolemy (Claudius) in about the year A.D. 75.

Thermodynamics, Laws of (1847, 1850, 1851, 1906)

The First and Second Laws were stated by R. Clausius in 1850, based on previous work, and were further developed by W. Thomson in 1851.

Zeroth Law of Thermodynamics. For systems in equilibrium, there is an intrinsic property: internal energy. Any two bodies or systems in equilibrium with a third body are in equilibrium with each other. A function of the state of a substance that takes on the same value for all substances in thermal equilibrium, which is the temperature. For closed systems, changes in the internal energy are:

$$dU = dQ - dW$$

where dU = change internal energy
dQ = heat transferred to system
W = external work

First Law of Thermodynamics (1842, 1847).* The total energy change of any system together with its surroundings is zero; also called the *Law of Conservation of Energy.* The energy, including that equivalent to mass, of the universe is constant. The First law was expressed by H. Helmholtz and R. Clausius, was based on work by J. von Mayer (1842), and is an extension of the work of J. Joule. The statement of the first law is:

$$\Delta U = Q + W$$

where ΔU = the change in internal energy of the system
Q = heat absorbed by the system
W = work done on the system

Second Law of Thermodynamics (1850). A general law of the natural tendency in which the entropy of the universe and of systems in the universe is tending to a maximum, also called *Law of Entropy.* All processes in nature tend to occur with an increase in entropy; the flow of heat is always from higher to lower temperature. Not all forms of energy are equally interchangeable, with other forms of energy tending to go to heat. The Carnot theorem, $\Delta S \geq 0$, published in 1824, provides a working equation embodying the principles of the second law, which was expressed by Lord Kelvin (William Thomson) and by R. Clausius, who coined the word *entropy.* L. Boltzmann provided the statistical foundation of the second law (1877).

Third Law of Thermodynamics (1906). Solutions and gases are excluded from the third law. The Nernst heat theorem, also identified as the third law of thermodynamics, was extended by Planck by adding the postulate that the absolute entropy of a pure solid or a pure liquid approaches zero at 0 K:

$$\lim_{T \to 0} S \to 0$$

The entropy is related to thermodynamic probability by:

$$S = k \ln W$$

where S = entropy
k = Boltzmann constant
W = statistical probability

Thus, the more random the molecules are arranged, the greater the values of W and S. For a completely ordered system, W = 1 and S = 1. An exception is for a crystalline structure, for which quantum theory shows that the entropy at 0° abs. is not zero, because the crystal may exist in more than one state and have entropy residues from nuclear spin.

Young Modulus, E

The Young modulus applies to an elastic material and is the ratio of unit stress to elastic strain, produced in tension or compression:

$$E = \Delta\sigma/\Delta\varepsilon$$

The Young modulus for aluminum is 10×10^6 psi; steel, 30×10^6 psi; wood, concrete (compression), 5000 psi (34.5 MPa). Elastic materials obey the Hooke law.

The Young modulus by stretching of a wire or rod is:

$$M = mgL/\pi r^2 e$$

where M = modulus, r = radius
m = mass, e = elongation
L = length

The modulus of rigidity for twisting of a bar is:

$$M = CL/\pi r^4 \theta$$

where C = couple, $C = mgx$, L = length
θ = twist, radians, r = radius

From Hall, C.W., *Laws and Models: Science, Engineering, and Technology,* CRC Press, Boca Raton, FL, 2000.

Types of Manufacturing — Characteristics and Examples

Volume	Very low	High	Highest
Variety	Highest	Low	Lowest
Flexibility	Highest	Low	Lowest
1. Job-shop production	Tool and die making Casting (foundry) Baking (bakery)		
2. Mass production		Auto assembly Bottling Apparel manufacturing	
3. Continuous production			Paper milling Refining Extrusion

From Schonberger, R.J., Types of manufacturing, in *The Technology Management Handbook,* Dorf, R.C., Ed., CRC Press, Boca Raton, FL, 1999, p. **13**-2.

Coefficient of Friction—Identical Metals*

Courtesy of Edmond E. Bisson and Donald H. Buckley

The following table gives coefficients of kinetic sliding friction for polycrystalline pure metals in contact with themselves.

	Coefficient of Friction			
	Lubricated		Unlubricated	
Metal	Oil or Grease†	Solid Film MoS_2	Dry-Sliding in Air	Vacuum, with Surfaces Cleaned
Body-Centered Cubic				
Iron on iron	0.15	.04–.08	1.0	Seizure (2)
Tantalum on tantalum	0.1	.04–.08	1.0	Seizure
Molybdenum on molybdenum	0.1	.04–.08	1.2	Seizure
Tungsten on tungsten	0.1	.04–.08	0.3	3.0 (3)
Chromium on chromium	0.34	.04–.08	0.4 (4)	—
Face-Centered Cubic				
Copper on copper	0.08	.04–.08	1.2–1.5	Seizure (6)
Nickel on nickel	0.28	.04–.08	0.8	Seizure
Silver on silver	0.55	.04–.08	1.5	Seizure
Gold on gold	0.2	.04–.08	2.0 (8)	Seizure
Aluminum on aluminum	0.12	.04–.08	1.0	Seizure
Platinum on platinum	0.25	.04–.08	1.2	Seizure
Rhodium on rhodium	0.1	.04–.08	0.4	3.0–5.0 (6)
Iridium on iridium	0.1	—	0.4	4.0
Lead on lead	0.1	—	2.0	—
Hexagonal				
Beryllium on beryllium	0.1	.04–.08	0.4	0.5 (6)
Magnesium on magnesium	0.08	.04–.08	0.4	0.6 (6)
Lanthanum on lanthanum	0.1	—	0.4 (6)	0.3 (6)
Titanium on titanium	0.1	.04–.08	0.6 (7)	1.2 (6)
Zirconium on zirconium	0.1	.04–.08	0.6	0.5 (6)
Rhenium on rhenium	0.1	.04–.08	0.4 (7)	0.3 (6)
Osmium on osmium	0.1	—	0.3	0.6 (6)
Ruthenium on ruthenium	0.1	—	0.3	0.5 (6)
Thallium on thallium	0.1	—	0.3 (6)	0.4 (6)
Cobalt on cobalt	0.1	.04–.08	0.5	0.4 (6)
Cadmium on cadmium	0.05	.04–.08	0.8 (4)	—
Zinc on zinc	0.04	—	0.9 (4)	—
Rhombohedral				
Bismuth	—	—	0.9 (4)	—

†Paraffinic oil plus 1% lauric acid.

Metal	Surface Film	Coefficient of Friction
Copper on copper	Oxide	0.8 (4)
	Sulfide	0.7 (4)
Steel on steel	Oxide	
	Fe_2O_3	0.6 (9)
	Fe_3O_4	0.4 (9)
	Sulfide (FeS)	0.5 (9)
	Chloride ($FeCl_2$)	0.1 (9)
	Oleic Acid	0.1 (9)
	Graphite	0.1 (9)
	Telfon (PTFE)	0.04 (4)

Coefficient of Friction—Identical Metals* (continued)

* Values are principally from Reference 1 and NASA data, except where indicated by other reference numbers in parentheses.

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Coefficient of Friction—Identical Alloy Pairs

Courtesy of Edmond E. Bisson and Donald H. Buckley

Coefficients of Kinetic Sliding Friction for Pairs of Identical Metal Alloys

	Coefficient of Friction*			
	Lubricated		Unlubricated	
Alloy	Oil or Grease†	Solid Film MoS_2	Dry-Sliding in Air	Vacuum, with Surface Cleaned
1020 steel on 1020 steel	0.1	.04–.08	0.5 (1)	Seizure
52100 steel on 52100 steel	0.1	.04–.08	0.5 (1)	5.0 (1)
440–C S.S. on 440–C S.S.	0.1	.04–.08	0.4 (1)	2.5
304 S.S. on 304 S.S.	0.1	.04–.08	0.9 (1)	Seizure
Cast iron on cast iron	0.1	.04–.08	0.3 (4)	—
M-1 tool steel on M-1 tool steel	0.1	.04–.08	0.5 (1)	—
Brass on brass	0.1 (2)	.04–.08(2)	0.4 (4)	—
Rene 41 on Rene 41	0.1	.04–.08	0.4 (1)	4.0
Inconel on Inconel	0.1	.04–.08	0.8 (1)	Seizure
Hastelloy D on Hastelloy D	0.1	—	0.7 (1)	Seizure
Cermet K 162 B on Cermet K 162 B	0.1	.04–.08	0.2 (1)	1.0
Stellite Star J on Stellite Star J	0.1	.04–.08	0.3 (1)	0.5 (1)
Co-25 Mo on Co-25 Mo	0.08	0.04	0.5 (3)	0.3
Ti-12 Sn on Ti-12 Sn	—	—	0.8 (1)	0.6 (3)
Ti-16 Al on Ti-16 Al	—	—	0.5 (3)	0.3 (3)

† Lubricated with a mineral oil containing oxidation and corrosion inhibitors.
* Data from NASA—Lewis Research Center, except where indicated by reference numbers in parentheses.

References

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Coefficient of Friction—Dissimilar Metals

Courtesy of Edmond E. Bisson and Donald H. Buckley

Following are coefficients of kinetic sliding friction for dissimilar pure metals in contact with each other.

	Coefficient of Friction*			
	Lubricated		Unlubricated	
Metal Couple	Oil or Grease†	Solid Film MoS_2	Dry-Sliding in Air	Vacuum, with Surface Cleaned
Aluminum on iron	0.1	.04–.08	1.1 (2)	Seizure
Aluminum on zinc	0.1 (1)	.04–.08	0.8 (2)	—
Cadmium on aluminum	0.1 (1)	.04–.08	0.6 (2)	—
Cadmium on bismuth	0.1 (1)	.04–.08	0.8 (2)	—
Cadmium on iron	0.1	.04–.08	0.6 (2)	—
Cadmium on zinc	0.1 (1)	.04–.08	0.6 (2)	—
Cobalt on iron	0.1	.04–.08	0.5 (2)	0.7 (3)
Cobalt on copper	0.1	.04–.08	0.9 (2)	—
Cobalt on aluminum	0.1 (1)	.04–.08	1.0 (2)	—
Copper on cadmium	0.1 (1)	.04–.08	0.9 (2)	—
Copper on zinc	0.1 (1)	.04–.08	0.9 (2)	—
Copper on iron	0.1	.04–.08	1.0 (2)	5.0 (1)
Copper on nickel	0.1	.04–.08	1.2 (4)	2.0 (4)
Copper on tungsten	0.1	.04–.08	0.4 (4)	0.5 (4)
Nickel on tungsten	0.1	.04–.08	0.3 (4)	4.0 (4)
Zinc on iron	0.1 (1)	.04–.08	0.9 (2)	—
Zinc on antimony	0.1 (1)	.04–.08	0.9 (2)	—
Zinc on bismuth	0.1 (1)	.04–.08	0.7 (2)	—

†Lubricated with mineral oil containing oxidation and corrosion inhibitors.

	Coefficient of Friction*	
Material Combination	Dry-Sliding	Boundary Lubrication‡
Hard steel on Babbitt (ASTM 1)	0.33 (6)	0.16 (6)
Hard steel on Babbitt (ASTM 8)	0.35 (6)	0.14 (6)
Hard steel on Babbitt (ASTM 10)	—	0.13 (6)
Monel on SAE 52100 bearing steel	0.4 (5)	0.33 (5)
Beryllium copper on SAE 52100 bearing stgeel	0.8 (5)	0.10 (5)
Brass on SAE 52100 bearing steel	0.5 (5)	0.12 (5)
Bronze on SAE 52100 bearing steel	0.3 (5)	0.17 (5)
Gray cast iron on SAE 52100 bearing steel	0.6 (5)	0.29 (5)
Nodular iron on SAE 52100 bearing steel	0.5 (5)	0.17 (5)
Nichrome V on SAE 52100 bearing steel	0.3 (5)	0.13 (5)
24ST-aluminum on SAE 52100 bearing steel	0.3 (5)	0.17 (5)

‡ Paraffinic oil with oxidation and corrosion inhibitor.

* Values from NASA data, except where indicated by reference numbers in parentheses.

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Coefficient of Friction—Single Crystals*

Courtesy of Edmond E. Bisson and Donald H. Buckley

The following tables give coefficients of kinetic sliding friction for single crystals—metals and non-metals.

Metals

Metal	Atomic Plane	Coefficient of Friction: In Air, 20˚C	Coefficient of Friction: In Vacuum, Clean
Single crystal copper on single crystal copper	(100)	0.60	>40
	(110)	0.40	>40
	(111)	0.21	21.0
Single crystal cobalt on polycrystalline cobalt	(0001)	0.40	0.35
	(1010)	—	0.80
Single crystal magnesium on polycrystalline magnesium	(0001)	0.30	0.40
	(1010)	—	0.90
Single crystal rhenium on polycrystalline rhenium	(0001)	0.20	0.29
	(1010)	0.25	0.38
Single crystal beryllium on polycrystalline beryllium	(0001)	0.45	0.48
	(1010)	0.46	0.51
Single crystal titanium on polycrystalline titanium	(0001)	0.48	0.56
	(1010)	0.25	0.36
Single crystal tungsten on single crystal tungsten	(100)	0.60	3.0 (2)
	(110)	0.41	1.9 (2)
	(210)	0.40	1.3 (2)

Non-Metals

Material	Atomic Plane and Direction		Coefficient of Friction: In Air, 20˚C	Coefficient of Friction: In Vacuum, Clean
Diamond on diamond	(100)	<100>	0.15 (3,4)	—
	(100)	<110>	0.05 (3,4)	—
	(111)	—	0.05 (3,4)	0.9 (4)
Sapphire on sapphire	(0001)	$\langle 11\bar{2}0 \rangle$	0.15	0.50
	(0001)	$\langle 10\bar{1}0 \rangle$	—	0.96
	$(10\bar{1}0)$	<1120>	0.20	0.93
	$(10\bar{1}0)$	<0001>	—	1.00
Diamond on magnesium oxide	(100)	<100>	0.07	—
Diamond on lithium fluoride	(100)	<110>	0.24	0.80
Diamond on potassium fluoride	(100)	<110>	0.71	—
Diamond on sodium chloride	(100)	<110>	0.47–0.70	—
Diamond on potassium bromide	(100)	<110>	0.85	—

* Data from Reference 1 unless otherwise indicated by reference numbers in parentheses.

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Coefficient of Friction—Non-Metals*

Courtesy of Edmond E. Bisson and Donald H. Buckley

Listed below are coefficients of kinetic sliding friction for plastics and other non-metals in identical pairs and on steel.

	Coefficient of Friction			
	Dry-Sliding in Air		Vacuum, with Clean Surface	
Material	On Itself	On Steel	On Itself	On Steel
Teflon (PTFE)	0.1	0.04	—	.2–.3 (7)
Nylon	0.15–0.25	0.2	—	—
Perspex	0.8	0.5	—	—
Polystyrene	0.5	0.3	—	—
PCFE	0.2	0.08	—	0.3 (7)
Polyimide	—	0.25 (7)	0.5 (7)	0.2 (7)
Bakelite	0.3	0.30 (8)	—	—
Titanium carbide	0.2	0.5	0.9	—
Glass	1.0	0.6	—	—
Diamond	0.1 (2)	0.1	0.9	—
Sapphire	0.2 (3)	0.15 (4)	0.8 (5)	0.2 (5)
Mica	1.0	—	—	—
Carbon	0.2	0.15	—	0.4 (6)
Graphite	0.1	0.1	0.8	0.3 (6)

* Data from Reference 1 unless otherwise indicated by reference numbers in parentheses.

References

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Coefficient of Friction—Lubricating Powders

Courtesy of Edmond E. Bisson and Donald H. Buckley

Listed below are coefficients of kinetic sliding friction for steel on steel (SAE 4620 on SAE 1020) with various powders between the surfaces.

Powder	Coefficient of Friction†	Powder	Coefficient of Friction†
Cadmium iodide, CdI_2	0.06	Zinc stearate, Zn $(C_{18}H_{35}O_2)_2$	0.11
Cadmium chloride, $CdCl_2$	0.07	Cobalt chloride, $CoCl_2$	0.10
Tungsten disulfide, WS_2	0.08	Mercury iodide, HgI_2	0.18
Silver sulfate, Ag_2SO_4	0.14	Copper bromide, $CuBr_2$	0.06
Lead iodide, PbI2	0.28	Silver iodide, AgI	0.25

† Data compiled from: "Advanced Bearing Technology," E.E. Bisson and W.J. Anderson, NASA SP-38, 1964.

From Bolz, R.E. and Tuve, G.L., Friction and lubrication, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, p. 614.

Coefficients of Static and Sliding Friction

Reference letters indicate the lubricant used; numbers in parentheses give sources (see References).

Key to Lubricants Used;

a = oleic acid
b = Atlantic spindle oil (light mineral)
c = caster oil
d = lard oil
e = Atlantic spindle oil plus 2% oleic acid
f = medium mineral oil
g = medium mineral oil plus 1/2% oleic acid
h = stearic acid
i = grease (zinc oxide base)
j = graphite
k = turbine oil plus 1% graphite
l = turbine oil plus 1% stearic acid
m = turbine oil (medium mineral)
n = olive oil
p = palmitic acid
q = ricinoleic acid
r = dry soap
s = lard
t = water
u = rape oil
v = 3-in-1 oil
w = octyl alcohol
x = triolein
y = 1% lauric acid in paraffin oil

Materials	Static		Sliding	
	Dry	Greasy	Dry	Greasy
Hard steel on hard steel	0.78 (1)	0.11 (1, *a*)	0.42 (2)	0.029 (5, *h*)
		0.23 (1, *b*)		0.081 (5, *c*)
		0.15 (1, *c*)		0.080 (5, *i*)
		0.11 (1, *d*)		0.058 (5, *j*)
		0.0075 (18, *p*)		0.084 (5, *d*)
		0.0052 (18, *h*)		0.105 (5, *k*)
				0.096 (5, *l*)
				0.108 (5, *m*)
				0.12 (5, *a*)
Mild steel on mild steel	0.74 (19)		0.57 (3)	0.09 (3, *a*)
				0.19 (3, *u*)
Hard steel on graphite	0.21 (1)	0.09 (1, *a*)		
Hard steel on Babbitt (ASTM 1)	0.70 (11)	0.23 (1, *b*)	0.33 (6)	0.16 (1, *b*)
		0.15 (1, *c*)		0.06 (1, *c*)
		0.08 (1, *d*)		0.11 (1, *d*)
		0.085 (1, *e*)		

Coefficients of Static and Sliding Friction (continued)

Materials	Static Dry	Static Greasy	Sliding Dry	Sliding Greasy
Hard steel on Babbitt (ASTM 8)	0.42 (11)	0.17 (1, *b*)	0.35 (11)	0.14 (1, *b*)
		0.11 (1, *c*)		0.065 (1, *c*)
		0.09 (1, *d*)		0.07 (1, *d*)
		0.08 (1, *e*)		0.08 (11, *h*)
Hard steel on Babbitt (ASTM 10)		0.25 (1, *b*)		0.13 (1, *b*)
		0.12 (1, *c*)		0.06 (1, *c*)
		0.10 (1, *d*)		0.055 (1, *d*)
		0.11 (1, *e*)		
Mild steel on cadmium silver				0.097 (2, *f*)
Mild steel on phosphor bronze			0.34 (3)	0.173 (2, *f*)
Mild steel on copper lead				0.145 (2, *f*)
Mild steel on cast iron		0.183 (15, *c*)	0.23 (6)	0.133 (2, *f*)
Mild steel on lead	0.95 (11)	0.5 (1, *f*)	0.95 (11)	0.3 (11, *f*)
Nickel on mild steel			0.64 (3)	0.178 (3, *x*)
Aluminum on mild steel	0.61 (8)		0.47 (3)	
Magnesium on mild steel			0.42 (3)	
Magnesium on magnesium	0.6 (22)	0.08 (22, *y*)		
Teflon on Teflon	0.04 (22)			0.04 (22, *f*)
Teflon on steel	0.04 (22)			0.04 (22, *f*)
Tungsten carbide on tungsten carbide	0.2 (22)	0.12 (22, *a*)		
Tungsten carbide on steel	0.5 (22)	0.08 (22, *a*)		
Tungsten carbide on copper	0.325 (23)			
Tungsten carbide on iron	0.8 (23)			
Bonded carbide on copper	0.35 (23)			
Bonded carbide on iron	0.8 (23)			
Cadmium on mild steel			0.46 (3)	
Copper on mild steel	0.53 (8)		0.36 (3)	0.18 (17, *a*)
Nickel on nickel	1.10 (16)		0.53 (3)	0.12 (3, *w*)
Brass on mild steel	0.51 (8)		0.44 (6)	
Brass on cast iron			0.30 (6)	
Zinc on cast iron	0.85 (16)		0.21 (7)	
Magnesium on cast iron			0.25 (7)	
Copper on cast iron	1.05 (6)		0.29 (7)	
Tin on cast iron			0.32 (7)	
Lead on cast iron			0.43 (7)	
Aluminum on aluminum	1.05 (16)		1.4 (3)	
Glass on glass	0.94 (8)	0.01 (10, *p*)	0.40 (3)	0.09 (3, *a*)
		0.005 (10, *g*)		0.116 (3, *v*)
Carbon on glass			0.18 (3)	
Garnet on mild steel			0.39 (3)	
Glass on nickel	0.78 (8)		0.56 (3)	
Copper on glass	0.68 (8)		0.53 (3)	
Cast iron on cast iron	1.10 (16)		0.15 (9)	0.070 (9, *d*)
				0.064 (9, *n*)
Bronze on cast iron			0.22 (9)	0.077 (9, *n*)
Oak on oak (parallel to grain)	0.62 (9)		0.48 (9)	0.164 (9, *r*)
				0.067 (9, *s*)
Oak on oak (perpendicular)	0.54 (9)		0.32 (9)	0.072 (9, *s*)
Leather on oak (parallel)	0.61 (9)		0.52 (9)	
Cast iron on oak			0.49 (9)	0.075 (9, *n*)
Leather on cast iron			0.56 (9)	0.36 (9, *t*)
				0.13 (9, *n*)
Laminated plastic on steel			0.35 (12)	0.05 (12, *t*)
Fluted rubber bearing on steel				0.05 (13, *t*)

Coefficients of Static and Sliding Friction (continued)

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The Greek Alphabet

Greek Letter		Greek Name	English Equivalent	Greek Letter		Greek Name	English Equivalent
Α	α	Alpha	(ä)	Ν	ϖ	Nu	(n)
Β	β	Beta	(b)	Ξ	ξ	Xi	(ks)
Γ	γ	Gamma	(g)	Ο	ο	Omicron	(o)
Δ	δ	Delta	(d)	Π	π	Pi	(p)
Ε	ε	Epsilon	(e)	Ρ	ρ	Rho	(r)
Ζ	ζ	Zeta (z)	(z)	Σ	σ ς	Sigma	(s)
Η	η	Eta	(ā)	Τ	τ	Tau	(t)
Θ	θ	Theta	(th)	Υ	υ	Upsilon	(ü, o͞o)
Ι	ι	Iota	(ē)	Φ	φ	Phi	(f)
Κ	κ	Kappa	(k)	Χ	χ	Chi	(H)
Λ	λ	Lambda	(l)	Ψ	ψ	Psi	(ps)
Μ	μ	Mu	(m)	Ω	ω	Omega	(o)

From Bolz, R.E. and Tuve, G.L., Communication, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, p. 793.

Units and Their Conversion

Policy of this Edition

In each table in this handbook, the numerical values are preferably expressed in those units most commonly used by U.S. engineers working in the specific field, but SI metric units have also been added. In some cases two tables are given, one in English units, one in metric. In other tables parallel columns showing figures in both units are used, or the conversion factors are listed.

In a general engineering handbook complete consistency in units, abbreviations, and symbols is hardly possible, or even desirable. Such consistency would quickly defeat the objective of providing quick access to numbers of maximum immediate usefulness. Within each special field of engineering, the technical societies and industry associations have developed certain uniform practices and standards; if tables and data are given only in units that are foreign to these prevailing standards, convenience is sacrificed. In any case the practical demands of compilation and new typesetting costs, and the usual requirement of a copyright owner that reprinted material should not be changed, may well govern the units used in any given table.

The present edition of this handbook reflects the changes in abbreviations, symbols, and forms that are resulting from the efforts to reduce the diversity of practices from one specialty to another and from one nation to another. Recommendations of the International Organization for Standardization (ISO–R 1000) and of the "Metric Practice Guide," adopted by ASTM, NBS, APL, and others, have focused attention on the diversity of so-called standards.

Since the United States is the only major industrial nation that has not yet converted to metric units, some legal requirements in that direction are to be expected. It is now a contradiction to speak of the "English" system of units, and for some time to come U.S. engineers must accommodate to a wide use of conversions from one set of units to another. The extensive conversion tables that follow are offered with this expectation.

In spite of major efforts to unify engineering practices, there are many good reasons for retaining several means of expressing a physical quantity. For ease of learning and communication a descriptive name is better than one arbitrarily assigned, such as Hz for cps, celsius for centigrade, and torr for mm Hg; an opposite trend is prevalent at this time. Numerical scales directly related to the physical phenomena and to the method of their measurement have an advantage in the laboratory or field and will not soon be abandoned. Examples are barometric pressure in mm or in. of mercury, viscosity in seconds Saybolt, the calorie or the Btu, and even the "coefficients" of expansion, friction, diffusion, attenuation, and reflection. Symbols, abbreviations, and even the units themselves are not infrequently subject to change; note, for example, the now preferred *dB* in place of the well-established *db*; elimination of widely used abbreviations, such as kwh, cps, gpm, cc, and psi; and revised values for the second, the calorie, or the atomic weights. Users of this handbook are invited to call attention to places where consistency could be improved without sacrificing the objectives.

Of the many named units that might have more than one value, this book uses (unless otherwise stated) the thermochemical gram-calorie (4.184 J), the thermochemical Btu (1 054.35 J), the avoirdupois pound and ounce, the statue mile (5 280 ft), the short ton (2 000 lb), the U.S. liquid gallon (231 in.^{3}), and the electrical horsepower (746 W).

Rather than present a special and condensed table of engineering conversion factors, the editors have chosen to reprint the large table that has been developed over the years for the *Handbook of Chemistry and Physics*. Certain specialized conversion factors and tables have been included.

The Metric International System (SI)

Moves toward an international system of metric units are now following each other in quick succession, so a table of conversion factors for the most common units is given herewith. Perhaps the most definite are the moves toward the SI standards already initiated by the National Bureau of Standards, the various military services, the National Aeronautics and Space Administration, and other U.S. Government research groups. The American Society for Testing and Materials has declared in favor of SI units and will give other units only a secondary place in all newly issued ASTM Standards.[a] Other major engineering societies have committees to explore the adoption of SI units and are holding many meetings for discussion among members.

Whatever the decisions about converting to the metric system, the actual process will require many years, as can readily be seen from the experiences of other countries; in Great Britain, for example, even the single conversion to decimal monetary units and coinage moves very slowly. The practices and standards among the metric-system countries are far from uniform; no real international system exists among them.

Mere conversion of present U.S. specifications, drawings, tools, machines, and stock sizes, to equivalent metric units (so-called "soft" conversion) will not in any sense result in an "international" system. Instead, a "hard" conversion representing the abandonment of the 1/2-fractional system in favor of a 1/10-fractional system is necessary to attain the real advantages of the metric system. This means re-sizing of all round and sheet stock, lumber, bolts, screws, nails, wires, gears, containers, modules, and sub-assemblies, plus all the tools and machines related thereto. A long period of double-stocking must follow. The entire change is made the more difficult by the great penetration of U.S. products and materials into the markets of the world, e.g., airplanes and military equipment, production, and construction machinery. This is not to mention the problem of the individual engineer, technician, and user, who visualizes all his size relationships in inches

Units and Their Conversion (continued)

and feet and his weights in pounds. Realistically, more than one generation will be required for the educational conversion alone.

In presenting data in international standard metric units throughout this edition, the practices and forms used in the "Metric Practice Guide" have been carefully followed.[a] Certain conventions used in the "Metric Practice Guide" are not consistent with those originally adopted for this handbook, nor with ANSI standards. Special attention is directed to the following conventions:

1. For degrees Kelvin the degree symbol is omitted; for example, 50 K, not 50°K.
2. For multiplication a center point is used; for example, the unit of dynamic viscosity is abbreviated as $N{\cdot}s/m^2$, not $N\ s/m^2$ or $N \times s/m^2$.
3. Symbols for SI units are not capitalized unless the unit is derived from a proper name, as N for Sir Isaac Newton; however, *unabbreviated* units are not capitalized, such as newton, kelvin, hertz.

Conversion Factors to SI Standard Units

To Convert	To	Multiply by	To Convert	To	Multiply by
Acceleration			Power		
feet/second^2	meters/second^2	0.3048	Btu/second	watt	1054.350
Area			foot-pounds/ second	watt	1.355818
square feet	square meters	0.09290304			
Energy			horsepower	watt	746.
Btu (mean)	joule	1055.87	Pressure		
calorie (mean)	joule	4.19002	atmosphere	newtons/meter^2	101325.0
electron volt	joule	1.60210×10^{-19}	bar	newtons/meter^2	100000.
foot-pound	joule	1.355818	kilograms/cm^2	newtons/meter^2	98066.50
watthour	joule	3600.	pounds/in.^{2}	newtons/meter^2	6894.757
Force			torr (mm Hg. 0°C)	newtons/meter^2	133.322
dyne	Newton	0.00001			
kilogram	Newton	9.80665	Viscosity		
pound	Newton	4.448222	centipoise	newton-second/ meter^2	0.001
Length					
foot	meter	0.3048000	pounds/foot second	newton-second/ meter^2	1.488164
mil	meter	0.0000254			
mile (U.S. statute)	meter	1609.344	Volume		
			cubic foot	cubic meter	0.02831685
Mass			gallon (U.S. liquid)	cubic meter	0.003785412
pound	kilogram	0.4535924			
slug	kilogram	14.59390			
ton (2000 lb)	kilogram	907.1847			

[a] See "Metric Practice Guide," ASTM Standard E 380–70, American Society for Testing and Materials, 1970.

From Bolz, R.E. and Tuve, G.L., Units and conversion factors, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, pp. 803–804.

International System (SI) Metric Units

Basic Units–MKS

Length	meter	m	Electric current	ampere	A
Mass	kilogram	kg	Thermodynamic temperature	kelvin	K
Time	second	s	Luminous intensity	candela	cd

Derived Units

Property	Unis†	Abbreviations and Dimensions	
Acceleration	meter per second squared	m/s^2	
Activity (of radioactive source)	1 per second	s^{-1}	
Angular acceleration	radian per second squared	$rads/s^{-1}$	
Angular velocity	radian per second	rad/s	
Area	square meter	m^2	
Density	kilogram per cubic meter	kg/m^3	
Dynamic viscosity	newton-second per sq meter	$N·s/m^2$	
Electric capacitance	farad	F	(A·s/V)
Electric charge	coulomb	C	(A·s)
Electric field strength	volt per meter	V/m	
Electric resistance	ohm		(V/A)
Entropy	joule per kelvin	J/K	
Force	newton	N	$(kg·m/s^2)$
Frequency	hertz	hz	(s^{-1})
Illumination	lux	lx	(lm/m^2)
Inductance	henry	H	(V·s/A)
Kinematic viscosity	sq meter per second	m^2/s	
Luminance	candela per sq meter	cd/m^2	
Luminous flux	lumen	lm	(cd·sr)
Magnetomotive force	ampere	A	
Magnetic field strength	ampere per meter	A/m	
Magnetic flux	weber	Wb	(V·s)
Magnetic flux density	tesla	T	(Wb/m^2)
Power	watt	W	(J/s)
Pressure	newton per square meter	N/m^2	
Radiant intensity	watt per steradian	W/sr	
Specific heat	joule per kilogram kelvin	J/kg K	
Thermal conductivity	watt per meter kelvin	W/m K	
Velocity	meter per second	m/s	
Volume	cubic meter	m^3	
Voltage, potential difference, electromotive force	volt	V	(W/A)
Wave number	1 per meter	m^{-1}	
Work, energy, quantity of heat	joule	J	(N·m)

International System (SI) Metric Units (continued)

Prefix Names of Multiples and Submultiples of Units

Decimal Equivalent	Prefix	Pronunciation	Symbol	Exponential Expression
1,000,000,000,000	tera	tĕr′ȧ	T	10^{+12}
1,000,000,000	giga	jĭ′gȧ	G	10^{+9}
1,000,000	mega	mĕg′ȧ	M	10^{+6}
1,000	kilo	kĭl′ō	k	10^{+3}
100	hecto	hĕk′tō	h	10^{+2}
10	deka	dĕk′ȧ	da	10
0.1	deci	dĕs′ĭ	d	10^{-1}
0.01	centi	sĕnt′ĭ	c	10^{-2}
0.001	milli	mĭl′ĭ	m	10^{-3}
0.000 001	micro	mī′krō	μ	10^{-6}
0.000 000 001	nano	năn′ō	n	10^{-9}
0.000 000 000 001	pico	pĕ′kō	p	10^{-12}
0.000 000 000 000 001	femto	fĕm′tō	f	10^{-15}
0.000 000 000 000 000 001	atto	ăt′tō	a	10^{-18}

Definitions of the Most Important International System (SI) Units

The *ampere* (unit of electric current) is the constant current that, if maintained in two straight parallel conductors of infinite length, of negligible circular sections, and placed 1 meter apart in a vacuum, will produce between these conductors a force equal to 2×10^{-7} newton per meter of length.

The *candela* is the luminous intensity, in the direction of the normal, of a blackbody surface 1/600,000 square meter in area, at the temperature of solidification of platinum under a pressure of 101,325 newtons per square meter.

The *coulomb* (unit of quantity of electricity) is the quantity of electricity transported in 1 second by a current of 1 ampere.

The *ephemeris second* (unit of time) is exactly 1/31 556 925.974 7 of the tropical year of 1900, January, 0 days, and 12 hours ephemeris time.

The *fraud* (unit of electric capacitance) is the capacitance of a capacitor between the plates of which there appears a difference of potential of 1 volt when it is charged by a quantity of electricity equal to 1 coulomb.

The *henry* (unit of electric inductance) is the inductance of a closed circuit in which an electromotive force of 1 volt is produced when the electric current in the circuit varies uniformly at a rate of 1 ampere per second.

The *International Practical Kelvin Temperature Scale* of 1960 and the *International Practical Celsius Temperature Scale* of 1960 are defined by a set of interpolation equations based on the following reference temperatures:

	K	Deg C
Oxygen, liquid-gas equilibrium	90.18	–182.97
Water, solid-liquid equilibrium	273.15	0.00
Water, solid-liquid-gas equilibrium	273.16	0.01
Water, liquid-gas equilibrium	373.15	100.00
Zinc, solid-liquid equilibrium	692.655	419.505
Sulfur, liquid-gas equilibrium	717.75	444.6
Silver, solid-liquid equilibrium	1233.95	960.8
Gold, solid-liquid equilibrium	1336.15	1063.0

The *joule* (unit of energy) is the work done when the point of application of 1 newton is displaced a distance of 1 meter in the direction of the force.

The *kelvin* (unit of thermodynamic temperature) is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water. The decision was made at the 13th General Conference on Weights and Measures on October 13, 1967, that the name of the unit of thermodynamic temperature would be changed from *degree Kelvin* (symbol: °*K*) to *kelvin* (symbol: *K*). The name (*kelvin*) and symbol (*K*) are to be used for expressing temperature intervals. The former convention that expressed a temperature interval in *degrees Kelvin* or, abbreviated, *deg K* is dropped. However, the old designations are acceptable temporarily as alternatives to the new ones. One may also express temperature intervals in *degrees Celsius*.

International System (SI) Metric Units (continued)

The *kilogram* (unit of mass) is the mass of a particular cylinder of platinum iridium alloy, called the International Prototype Kilogram, which is preserved in a vault at Sèvres, France, by the International Bureau of Weights and Measures.

Length: The name *micron*, for a unit of length equal to 10^{-6} meter, and the symbol μ that has been used for it were dropped by action of the 13th General Conference on Weights and Measures on October 13, 1967. The symbol μ is to be used solely as an abbreviation for the prefix *micro*-, standing for the multiplication by 10^{-6}. Thus the length previously designated as 1 micron should be designated 1 μm.

The *lumin* (unit of luminous flux) is the luminous flux emitted in a solid angle of 1 steradian by a uniform point source having an intensity of 1 candela.

The *newton* (unit of force) is that force that gives to a mass of 1 kilogram an acceleration of 1 meter per second.

The *ohm* (unit of electric resistance) is the electric resistance between two points of a conductor when a constant difference of potential of 1 volt, applied between these two points, produces in this conductor a current of 1 ampere, this conductor not being the source of any electromotive force.

The *meter* (unit of length) is the length of exactly 1 650 763.73 wavelengths of the radiation in vacuum corresponding to the unperturbed transition between the levels $2p_{10}$ and $5d_5$ of the atom of krypton 86, the orange-red line.

The *second* is the unit of time of the International System of Units. The definition adopted at the October 13, 1967, meeting of the 13th General Conference on Weights and measures is: "The second is he duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the fundamental state of the atom of cesium 133." The frequency (9 192 631 770 hz), which the definition assigns to the cesium radiation, was carefully chosen to make it impossible, by any existing experimental evidence, to distinguish the new second from the *ephemeris second* based on the earth's motion. Therefore no changes need to be made in data stated in terms of the old standard in order to convert them to the new one. The atomic definition has two important advantages over the previous definition: (1) it can be realized (i.e., generated by a suitable clock) with sufficient precision, ± 1 part per hundred billion (10^{11}) or better, to meet the most exacting demands of modern metrology; and (2) it is available to anyone who has access to or who can build an atomic clock controlled by the specified cesium radiation.‡ In addition one can compare other high-precision clocks directly with such a standard in a relatively short time — an hour or so compared against years with the astronomical standard. Laboratory-type atomic clocks are complex and expensive, so that most clocks and frequency generators will continue to be calibrated against a standard such as the NBS Frequency Standard, controlled by a cesium atomic beam, at the Radio Standards Laboratory in Boulder, Colorado. In most cases the comparison will be by way of the standard-frequency and time-interval signals broadcast by NBS radio stations WWV, WWVH, WWVB, and WWVL.

The *volt* (unit of electric potential difference and electromotive force) is the difference of electric potential between two points of a conducting wire carrying a constant current of 1 ampere, when the power dissipated between these points is equal to 1 watt.

The *watt* (unit of power) is the power that gives rise to the production of energy at the rate of 1 joule per second.

The *weber* (unit of magnetic flux) is the magnetic flux that, linking a circuit of one turn, produces in it an electromotive force of 1 volt as it is reduced to zero at a uniform rate in 1 second.

† According to SI terminology, the following should be treated as obsolete:

angstrom (now 100 picometers or 0.1 nanometer)	liter (now cubic decimeter)
bar (now 100 kilonewtons/meter^2)	metric ton (now megagram)
kiloliter (now cubic meter)	micron (now micrometer)
kiloton (now gigagram)	

‡ A description of such clocks is given in "Atomic Frequency Standards," *NBS Tech. News Bull.*, 45:8–11, January 1961. For more developments and technical details, see R.E. Bechler, R.C. Mockler, and J.M. Richardson, "Cesium Beam Atomic Time and Frequency Standards," *Metrologia*, 1:114–131, July 1965.

From Bolz, R.E. and Tuve, G.L., Units and conversion factors, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, pp. 805–807.

International Metric System

This table can be used for conversion of any quantity in English units to corresponding SI units to give significant figures (without the use of a calculator). Exact values are shown in boldface. Unless otherwise stated, values are in thermochemical calorie, thermochemical Btu, and avoirdupois mass units.[a]

Instruction: Shift decimal as required for each digit in the original quantity and add the converted results.

Example: Convert an acceleration of 15.30 ft/s² to m/s².

Solution: From first line of table, 3.048 0 + 1.524 0 + 0.091 44 = 4.663 4 m/s².

	1	2	3	4	5	6	7	8	9
ACCELERATION									
foot/second² to meter/second², m/s²	**0.304 8**	0.609 6	0.914 4	1.219 2	1.524 0	1.828 8	2.133 6	2.438 4	2,743 2
g's (free fall, standard to meter/second², m/s²	**9.806 65**	19.613	29.420	39.227	49.033	58.840	68.647	78.453	88.260
inch/second² to meter/second², m/s²	**0.025 4**	0.050 8	0.076 2	0.101 6	0.127 0	0.152 4	0.177 8	0.203 2	0.228 6
AREA									
acre to meter², m²	4 046.856	8 093.7	12 141	16 187	20 234	24 281	28 328	32 375	36 422
circular mil to meter², m²	$5.067\ 075 \times 10^{-10}$	10.134×10^{-10}	15.201×10^{-10}	20.268×10^{-10}	25.335×10^{-10}	30.402×10^{-10}	35.470×10^{-10}	40.537×10^{-10}	45.604×10^{-10}
foot² to meter², m²	**0.092 903 04**	0.185 81	0.278 71	0.371 61	0.464 52	0.557 42	0.650 32	0.743 22	0.836 13
inch² to meter², m²	**0.000 645 16**	**0.001 290 32**	**0.001 935 48**	**0.002 580 64**	**0.003 225 80**	**0.003 870 96**	**0.004 516 12**	**0.005 161 28**	**0.005 806 44**
mile² (U.S. statute) to meter², m²	2 589 988	5 180 000	7 770 000	10 360 000	12 950 000	15 540 000	18 130 000	20 720 000	23 310 000
yard² to meter², m²	**0.836 127 36**	1.672 3	2.508 4	3.344 5	4.180 6	5.016 8	5.852 9	6.689 0	7.525 1
BENDING MOMENT OR TORQUE									
ounce-force-inch to newton-meter, N·m	0.007 061 552	0.014 123	0.021 185	0.028 246	0.035 308	0.042 369	0.049 431	0.056 492	0.063 554
pound-force-inch to newton-meter, N·m	0.112 984 8	0.225 97	0.338 95	0.451 94	0.564 92	0.677 91	0.790 89	0.903 88	1.016 9
pound-force-foot to newton-meter, N·m	1.355 818	2.711 6	4.067 5	5.423 3	6.779 1	8.134 9	9.490 7	10.847	12.202
DENSITY (MASS/VOLUME)									
grain/gallon to kilogram/meter³, kg/m³	0.017 118 06	0.034 236	0.051 354	0.068 472	0.085 590	0.102 71	0.119 83	0.136 94	0.154 06
ounce/gallon to kilogram/meter³, kg/m³	7.489 152	14.978	22.467	29.957	37.446	44.935	52.424	59.913	67.402

ounce/inch^3 to kilogram/meter^3, kg/m^3	1 729.994	3 460.0	5 190.0	6 920.0	8 650.0	10 380	12 110	13 840	15 570
pound-mass/foot^3 to kilogram/meter^3, kg/m^3	16.018 46	32.037	48.055	64.074	80.092	96.111	112.13	128.15	144.17
pound-mass/inch^3 to kilogram/meter^3, kg/m^3	27 679.90	55 360	83 040	110 720	138 400	166 080	193 760	221 440	249 120
pound-mass/gallon to kilogram/meter^3, kg/m^3	119.826 4	239.65	359.48	479.31	599.13	718.96	838.78	958.61	1 078.4
slug/foot^3 to kilogram/meter^3, kg/m^3	515.378 8	1 030.8	1 546.1	2 061.5	2 576.9	3 092.3	3 607 7	4 123.0	4 638.4
ELECTRICITY AND MAGNETISM									
ampere-hour to coulomb, C	**3 600**	**7 200**	**10 800**	**14 400**	**18 000**	**21 600**	**25 200**	**28 800**	**32 400**
faraday (based on C-12) to coulomb, C	96 487.00	192 970	289 460	385 950	482 440	578 920	675 410	771 900	868 380
gauss to tesla, T	**0.000 1**	**0.000 2**	**0.000 3**	**0.000 4**	**0.000 5**	**0.000 6**	**0.000 7**	**0.000 8**	**0.000 9**
gilbert to ampere-turn	0.795 774 7	1.591 5	2.387 3	3.183 1	3.978 9	4.774 6	5.570 4	6.366 2	7.162.0
oersted to ampere-meter, A/m	79.577 47	159.15	238.73	318.31	397.89	477.46	557.04	636.62	716.20
unit pole to weber, Wb	1.256 637 × 10^{-7}	2.513 3 × 10^{-7}	3.769 9 × 10^{-7}	5.026 5 × 10^{-7}	6.283 2 × 10^{-7}	7.539 8 × 10^{-7}	8.796 5 × 10^{-7}	10.053 × 10^{-7}	11.310 × 10^{-7}
ENERGY AND WORK									
British thermal unit to joule, J	1 054.350	2 108.7	3 163.1	4 217.4	5 271.8	6 326.1	7 380.5	8 434.8	9 489.2
British thermal unit (IT) to joule, J[a]	1 055.056	2 220.1	3 165.2	4 220.2	5 275.3	6 330.3	7 385.4	8 440.4	9 495.5
calorie to joule, J	**4.184**	**8.368**	**12.552**	**16.736**	**20.920**	**25.104**	**29.288**	**33.472**	**37.656**
calorie (IT) to joule, J[a]	**4.186 8**	**8.373 6**	**12.560 4**	**16.747 2**	**20.934 0**	**25 120 8**	**29.307 6**	**33.494 4**	**37.681 2**
electron volt to joule, J	1.602 10 × 10^{-19}	3.204 2 × 10^{-19}	4.806 3 × 10^{-19}	6.408 4 × 10^{-19}	8.010 5 × 10^{-19}	9.612 6 × 10^{-19}	11.215 × 10^{-19}	12.817 × 10^{-19}	14.419 × 10^{-19}
foot-pound-force to joule, J	1.355 818	2.711 6	4.067 5	5.423 3	6.779 1	8.134 9	9.490 7	10.847	12.202
kilowatt-hour to joule, J	**3 600 000**	**7 200 000**	**10 800 000**	**14 400 000**	**18 000 000**	**21 600 000**	**25 200 000**	**28 800 000**	**32 400 000**
horsepower-hour to joule, J	2 684 520	5 369 039	8 053 559	10 738 078	13 422 598	16 107 117	18 791 637	21 476 156	24 160 676
FLOW RATE									
foot^3/minute to meter^3/second, m^3/s	0.000 471 947 4	0.000 943 89	0.001 415 8	0.001 887 8	0.002 359 7	0.002 831 7	0.003 303 6	0.003 775 6	0.004 247 5
foot^3/second to meter^3/second, m^3/s	0.028 316 85	0.056 634	0.084 951	0.113 27	0.141 58	0.169 90	0.198 22	0.226 53	0.254 85
gallon (U.S. liquid)/day to meter^3/second, m^3/s	4.381 264 × 10^{-8}	8.762 × 10^{-8}	13.144 × 10^{-8}	17.525 × 10^{-8}	21.906 × 10^{-8}	26.288 × 10^{-8}	30.669 × 10^{-8}	35.050 × 10^{-8}	39.431 × 10^{-8}

Conversions to SI Units (continued)

	1	2	3	4	5	6	7	8	9
gallon (U.S. liquid)/minute to meter³/second, m³/s	0.000 063 090 20	0.000 126 18	0.000 189 27	0.000 252 36	0.000 315 45	0.000 378 54	0.000 441 63	0.000 504 72	0.000 567 81
pound-mass/hour to kilogram/second, kg/s	0.000 125 997 9	0.000 252 00	0.000 377 99	0.000 503 99	0.000 629 99	0.000 755 99	0.000 881 99	0.001 007 98	0.001 133 98
pound-mass/minute to kilogram/second, kg/s	0.007 559 873	0.015 120	0.022 680	0.030 239	0.037 799	0.045 359	0.052 919	0.060 479	0.068 039
FORCE									
kilogram-force to newton, N	**9.806 65**	19.613	29.420	39.227	49.033	58.840	68.647	78.453	88.260
ounce-force to newton, N	0.278 014 0	0.556 03	0.834 04	1.112 1	1.390 1	1.668 1	1.946 1	2.224 1	2.502 1
pound-force to newton, N	4.448 222	8.896 4	13.345	17.793	22.241	26.689	31.138	35.586	40.034
HEAT									
SPECIFIC HEAT CAPACITY									
British thermal unit/ pound-mass-deg F to joule/kilogram-kelvin, J/kg·K	**4 184**	**8 368**	**12 552**	**16 736**	**20 920**	**25 104**	**29 288**	**33 472**	**37 656**
British thermal unit (IT)/ pound-mass-deg F to joule/kilogram-kelvin, J/kg·K[a]	**4 186.8**	**8 373.6**	**12 560.4**	**16 747.2**	**20 934.0**	**25 120.8**	**29 307.6**	**33 494.4**	**37 681.2**
calorie/gram-deg C to joule/ kilogram-kelvin, J/kg·K	**4 184**	**8 368**	**12 552**	**16 736**	**20 920**	**25 104**	**29 288**	**33 472**	**37 656**
ENERGY/MASS (ENTHALPY, ETC.)									
British thermal unit/ pound-mass to joule/ kilogram, J/kg	2 324.444	4 648.9	6 973.3	9 287.8	11 622	13 947	16 271	18 596	20 920
British thermal unit (IT)/ pound-mass to joule/ kilogram, J/kg[a]	**2 326**	**4 652**	**6 978**	**9 304**	**11 630**	**13 956**	**16 282**	**18 608**	**20 934**
calorie/gram to joule/ kilogram, J/kg	**4 184**	**8 368**	**12 552**	**16 736**	**20 920**	**25 104**	**29 288**	**33 472**	**37 656**

THERMAL CONDUCTIVITY									
British thermal unit/hour-foot-deg F to watt/meter-kelvin, W/m·K	1.729 577	3.459 2	5.188 7	6.918 3	8.647 9	10.377	12.107	13.837	15.566
British thermal unit (IT)/hour-foot-deg F to watt/meter-kelvin, W/m·K[a]	1.730 735	3.461 5	5.192 2	6.922 9	8.653 7	10.384	12.115	13.846	15.577
British thermal unit-inch/hour-foot²-deg F to watt/meter-kelvin, W/m·K	0.144 131 4	0.288 26	0.432 39	0.576 53	0.720 66	0.864 79	1.008 9	1.153 1	1.297 2
British thermal unit (IT)-inch/hour-foot²-deg F to watt/meter-kelvin, W/m·K[a]	0.144 227 9	0.288 46	0.432 68	0.576 91	0.721 14	0.865 37	1.009 60	1.153 82	1.298 05
calorie second-centimeter-deg C to watt/meter-kelvin, W/m·K	**418.4**	**836.8**	**1 255.2**	**1 673.6**	**2 092.0**	**2 510.4**	**2 928.8**	**3 347.2**	**3 765.6**
ENERGY PER UNIT AREA									
British thermal unit/foot² to joule/meter², J/m²	11 348.93	22 698	34 047	45 396	56 745	68 094	79 443	90 791	102 140
calorie/centimeter² to joule/meter², J/m²	**41 840**	**836 80**	**125 520**	**167 360**	**209 200**	**251 040**	**292 880**	**334 720**	**376 560**
THERMAL DIFFUSIVITY									
foot²/hour to meter²/second, m²/s	**0.000 025 806 4**	**0.000 051 612 8**	**0.000 077 419 2**	**0.000 103 256**	**0.000 129 032**	**0.000 154 838 4**	**0.000 180 644 8**	**0.000 206 451 2**	**0.000 232 257 6**
THERMAL RESISTANCE									
deg F-hour foot²/British thermal unit to kelvin-meter²/watt, K·m²/W	0.176 228 0	0.352 46	0.528 68	0.704 91	0.881 14	1.057 4	1.233 6	1.409 8	1.586 1
THERMAL CONDUCTANCE									
British thermal unit/hour-foot²-deg F to watt/meter²-kelvin, W/m²·K	5.674 466	11.349	17.023	22.698	28.372	34.047	39.721	45.396	51.070
British thermal unit/second-foot²-deg F to watt/meter²-kelvin, W/m²·K	20 428.08	40 856	61 284	81 712	102 140	122 570	143 000	163 420	183 850

Conversions to SI Units (continued)

	1	2	3	4	5	6	7	8	9
calorie/second-centimeter²-deg C to watt/meter²-kelvin, W/m·K	**41 840**	**83 680**	**125 520**	**167 360**	**209 200**	**251 040**	**292 880**	**334 720**	**376 560**
LENGTH									
caliber to meter, m	**0.000 254**	**0.000 508**	**0.000 762**	**0.001 016**	**0.001 270**	**0.001 524**	**0.001 778**	**0.002 032**	**0.002 286**
fathom to meter, m	**1.828 8**	**3.657 6**	**5.486 4**	**7.315 2**	**9.144 0**	**10.972 8**	**12.801 6**	**14.630 4**	**16.459 2**
foot to meter, m	**0.304 8**	**0.609 6**	**0.914 4**	**1.219 2**	**1.524 0**	**1.828 8**	**2.133 6**	**2.438 4**	**2.743 2**
inch to meter, m	**0.025 4**	**0.050 8**	**0.076 2**	**0.101 6**	**0.127 0**	**0.152 4**	**0.177 8**	**0.203 2**	**0.228 6**
light year to meter, m	$9.460\ 550 \times 10^{15}$	18.921×10^{15}	28.382×10^{15}	37.842×10^{15}	47.303×10^{15}	56.763×10^{15}	55.224×10^{15}	75.684×10^{15}	85.145×10^{15}
mil to meter, m	**0.000 025 4**	**0.000 050 8**	**0.000 076 2**	**0.000 101 6**	**0.000 127 0**	**0.000 152 4**	**0.000 177 8**	**0.000 203 2**	**0.000 228 6**
mile (U.S. nautical) to meter, m	**1 852**	**3 704**	**5 556**	**7 408**	**9 260**	**11 112**	**12 964**	**14 816**	**16 668**
mile (U.S. statute) to meter, m	**1 609.344**	3 218.7	4 828.0	6 437.4	8 046.7	9 656.1	11 265	12 875	14 484
rod to meter, m	**5.029 2**	**10.058 4**	**15.087 6**	**20.116 8**	**25.146 0**	**30.175 2**	**35.204 4**	**40.233 6**	**45.262 8**
yard to meter, m	**0.914 4**	**1.828 8**	**2.743 2**	**3.657 6**	**4.572 0**	**5.486 4**	**6.400 8**	**7.315 2**	**8.229 6**
MASS									
grain to kilogram, kg	**0.000 064 798 91**	0.000 129 60	0.000 194 40	0.000 259 20	0.000 324 00	0.000 388 80	0.000 453 60	0.000 518 40	0.000 583 20
ounce-mass to kilogram, kg	0.028 349 52	0.056 699	0.085 049	0.113 40	0.141 75	0.170 10	0.198 45	0.226 80	0.255 15
ounce-mass (troy or apothecary) to kilogram, kg	0.031 103 48	0.062 207	0.093 310	0.124 41	0.155 52	0.186 62	0.217 72	0.248 83	0.279 93
pound-mass to kilogram, kg	**0.453 592 37**	0.907 18	1.360 8	1.814 4	2.268 0	2.721 6	3.175 1	3.628 7	4.082 3
pound-mass (troy or apothecary) to kilogram, kg	0.373 241 7	0.746 48	1.119 7	1.493 0	1.866 2	2.239 5	2.612 7	2.985 9	3.359 2
slug to kilogram, kg	14.593 90	29.188	43.782	58.376	72.970	87.563	102.16	116.75	131.35
ton (long, 2 240 lb_m) to kilogram, kg	1 016.047	2 032.1	3 048.1	4 064.2	5 080.2	6 096.3	7 112.3	8 128.4	9 144.4
ton (short, 2 000 lb_m) to kilogram, kg	907.184 7	1 814.4	2 721 6	3 628.7	4 535.9	5 443.1	6 350.3	7 257.5	8 164.7

POWER									
British thermal unit/second to watt, W	1 0543.350	2 108.7	3 163.1	4 217.4	5 271.8	6 326.1	7 380.5	8 434.8	9 489.2
British thermal unit/minute to watt, W	17.572 50	35.145	52.718	70.290	87.863	105.44	123.01	140.58	158.15
British thermal unit/hour to watt, W	0.292 875 1	0.585 75	0.878 63	1.171 5	1.464 4	1.757 3	2.050 1	2.343 0	2.635 9
British thermal unit (IT)/ hour to watt, W[a]	0.293 071 1	0.586 14	0.879 21	1.172 3	1.465 4	1.758 4	2.051 5	2.344 6	2.637 6
calorie/second to watt, W	**4.184**	**8.368**	**12.552**	**16.736**	**20.920**	**25.104**	**29.288**	**33.472**	**37.656**
calorie/minute to watt, W	0.069 733 33	0.139 47	0.209 20	0.278 93	0.348 67	0.418 40	0.488 13	0.557 87	0.627 60
foot-pound-force/second to watt, W	1.355 818	2.711 6	4.067 5	5.423 3	6.779 1	8.134 9	9.490 7	10.847	12 202
foot-pound-force/minute to watt, W	0.022 596 97	0.045 194	0.067 791	0.090 388	0.112 98	0.135 58	0.158 18	0.180 78	0.203 37
foot-pound-force/hour to watt, W	0.,000 376 616 1	0.000 753 23	0.001 129 8	0.001 506 5	0.001 883 1	0.002 259 7	0.002 636 3	0.003 012 9	0.003 389 5
horsepower (550 ft·lb$_f$/s) to watt, W	745.699 9	1 491.4	2 237.1	2 982.8	3 728.5	4 474.2	5 219.9	5 965.6	6 711.3
horsepower (electric) to watt, W	**746.**	**1 492.**	**2 238.**	**2 984.**	**3 730.**	**4 476.**	**5 222.**	**5 968.**	**6 714.**
tons of refrigeration to watt, W	3 516.853	7 033.7	10 551	14 067	17 584	21 101	24 618	28 135	31 652
POWER/AREA									
British thermal unit/foot^2-second to watt/meter^2, W/m^2	11 348.93	22 698	34 047	45 396	56 745	68 094	79 443	90 791	102 140
British thermal unit/foot^2-minute to watt/meter^2, W/m^2	189.148 9	378.30	567.45	756.60	945.74	1 134.9	1 324.0	1 513.2	1 702.3
British thermal unit/foot^2-hour to watt/meter^2, W/m^2	3.152 481	6.305 0	9.457 4	12.610	15.762	18.915	22.067	25.220	28.372
British thermal unit/inch^2-second to watt/meter^2, W/m^2	1 634 246	3 268 500	4 902 700	6 537 000	8 171 200	9 805 500	11 440 000	13 074 000	14 708 000
calorie/centimeter^2-minute to watt/meter^2, W/m^2	697.333 3	1 394.7	2 092.0	2 789.3	3 486.7	4 184.0	4 881.3	5 578.7	6 276.0

Conversions to SI Units (continued)

	1	2	3	4	5	6	7	8	9
PRESSUSRE OR STRESS (FORCE/AREA)									
atmosphere (normal = 760 torr) to newton/meter², N/m²	**101 325**	**202 650**	**303 975**	**405 300**	**506 625**	**607 950**	**709 275**	**810 600**	**911 925**
bar to newton/meter², N/m²	**100 000**	**200 000**	**300 000**	**400 000**	**500 000**	**600 000**	**700 000**	**800 000**	**900 000**
foot of water (39.2 F) to newton/meter², N/m²	2 988.980	5 978.0	8 966.9	11 956	14 945	17 934	20 923	23 912	26 901
inch of mercury (32 F) to newton/meter², N/m²	3 386.389	6 772.8	10 159	13 546	16 932	20 318	23 705	27 091	30 478
inch of water (39.2 F) to newton/meter², N/m²	249.082 0	498.16	747.25	996.33	1 245.4	1 494.5	1 743.6	1 992.7	2 241.7
inch of water (60 F) to newton/meter², N/m²	248.840 0	497.68	746.52	995.36	1 244.2	1 493.0	1 741.9	1 900.7	2 239.6
kilogram-force/centimeter² to newton/meter², N/m²	**98 006.5**	**196 133**	**294 199.5**	**392 266**	**490 332.5**	**588 399**	**686 465.5**	**784 532**	**882 598.5**
millimeter of mercury (0 C), torr, to newton/meter², N/m²	133.322 4	266.64	399.97	533.29	666.61	799.93	933.26	1 066.6	1 199.9
pound-force/foot² to newton/meter², N/m²	47.880 26	95.761	143.64	191.52	239.40	287.28	335.16	383.04	430.92
pound-force/inch² (psi) to newton/meter², N/m²	6 894.757	13 790	20 684	27 579	34 474	41 369	48 263	55 158	62 053
VELOCITY									
foot/hour to meter/second, m/s	0.000 084 666 67	0.000 169 33	0.000 254 00	0.000 338 67	0.000 423 33	0.000 508 00	0.000 592 67	0.000 677 33	0.000 762 00
foot/minute to meter/second, m/s	**0.005 08**	**0.010 16**	**0.015 24**	**0.020 32**	**0.025 40**	**0.030 48**	**0.035 56**	**0.040 64**	**0.045 72**
foot/second to meter/second, m/s	**0.304 8**	**0.609 6**	**0.914 4**	**1.219 2**	**1.524 0**	**1.828 8**	**2.133 6**	**2.438 4**	**2.743.2**
inch/second to meter/second, m/s	**0.025 4**	**0.050 8**	**0.076 2**	**0.101 6**	**0.127 0**	**0.152 4**	**0.177 8**	**0.203 2**	**0.228 6**
kilometer/hour to meter/second, m/s	0.277 777 8	0.555 56	0.833 33	1.111 1	1.388 9	1.666 7	1.944 4	2.222 2	2.500 0

knot (international) to meter/second, m/s	0.514 444 4	1.028 9	1.543 3	2.057 8	2.572 2	3.086 7	3.601 1	4.115 6	4.630 0
mile/hour (U.S. statute) to meter/second, m/s	**0.447 04**	**0.894 08**	**1.341 12**	**1.788 16**	**2.235 20**	**2.682 24**	**3.129 28**	**3.576 32**	**4.023 36**
mile/minute (U.S. statute) to meter/second, m/s	**26.822 4**	**53.644 8**	**80.467 2**	**107.289 6**	**134.112 0**	**160.934 4**	**187.756 8**	**214.587 92**	**241.401 6**
mile/second (U.S. statute) to meter/second, m/s	**1 609.344**	3 218.7	4 828.0	6 437.4	8 046.7	9 656.1	11 265	12 875	14 484
VISCOSITY									
DYNAMIC OR ABSOLUTE, μ									
centipoise to newton-second/meter^2, N·s/m^2	**0.001**	**0.002**	**0.003**	**0.004**	**0.005**	**0.006**	**0.007**	**0.008**	**0.009**
pound-mass/foot-second to newton-second/meter^2, N·s/m^2	1.488 164	2.976 3	4.464 5	5.952 7	7.440 8	8.929 0	10.417	11.905	13.393
pound-force-second/foot^2 to newton-second/meter^2, N·s/m^2	47.880 26	95.761	143.64	191.52	239.40	287.28	335.16	383.04	430.92
slug/foot-second to newton-second/meter^2, N·s/m^2	47.880 26	95.761	143.64	191.52	239.40	287.28	335.16	383.04	430.92
KINEMATIC, ν									
centistoke to meter^2/second, m^2/s	$1. \times 10^{-6}$	$2. \times 10^{-6}$	$3. \times 10^{-6}$	$4. \times 10^{-6}$	$5. \times 10^{-6}$	$6. \times 10^{-6}$	$7. \times 10^{-6}$	$8. \times 10^{-6}$	$9. \times 10^{-6}$
foot^2/second to meter^2/second, m^2/s	**0.092 903 04**	0.185 81	0.278 71	0.371 61	0.464 52	0.557 42	0.650 32	0.743 22	0.836 12
VOLUME									
acre-foot to meter^3, m^3	1 233.482	2 467.0	3 700.4	4 933.9	6 167.4	7 400.9	8 634.4	9 867.9	11 101
barrel (oil, 42 gal) to meter^3, m^3	0.158 987 3	0.217 97	0.476 96	0.635 95	0.794 94	0.953 92	1.112 9	1.271 9	1.430 9
board foot to meter^3, m^3	0.002 359 737	0.004 719 5	0.007 079 2	0.009 438 9	0.011 799	0.141 58	0.016 518	0.018 878	0.021 238
bushel (U.S.) to meter^3, m^3	0.035 239 07	0.070 478	0.105 72	0.140 96	0.176 20	0.211 43	0.246 67	0.281 91	0.317 15
foot^3 to meter^3, m^3	0.028 316 85	0.056 634	0.084 951	0.113 27	0.141.58	0.169 90	0.198 22	0.226 53	0.254 85
gallon (U.S. liquid) to meter^3, m^3	0.003 785 412	0.,007 570 8	0.011 356	0.015 142	0.018 927	0.022 712	0.026 498	0.030 283	0.034 069
inch^3 to meter^3, m^3	0.000 016 387 06	0.000 032 774	0.000 049 161	0.000 065 548	0.000 081 935	0.000 098 322	0.000 114 71	0.000 131 10	0.000 147 48

Conversions to SI Units (continued)

	1	2	3	4	5	6	7	8	9
ounce (U.S. fluid) to meter^3, m^3	0.000 029 573 53	0.000 059 147	0.000 088 721	0.000 118 29	0.000 147 87	0.000 177 44	0.000 207 01	0.000 236 59	0.000 266 16
peck (U.S.) to meter^3, m^3	0.008 809 768	0.017 620	0.026 429	0.035 239	0.044 049	0.052 859	0.061 668	0.070 478	0.079 288
quart (U.S. liquid) to meter^3, m^3	0.000 946 352 9	0.001 892 7	0.002 839 1	0.003 785 4	0.004 731 8	0.005 678 1	0.006 624 5	0.007 570 8	0.008 517 2
yard^3 to meter^3, m^3	0.764 554 9	1.529 1	2.293 7	3.058 2	3.822 8	4.587 3	5.351 9	6.116 4	6.881 0
VOLUME/MASS (SPECIFIC VOLUME)									
foot^3/pound to meter^3/kilogram, m^3/kg	0.062 427 96	0.124 86	0.187 28	0.249 71	0.312 14	0.374 57	0.437 00	0.499 42	0.561 85

[a] The thermochemical calorie is exactly 4.184 joules by definition. The international steam table (IT) calories is exactly 4.186 8 joules by definition. The thermochemical Btu is 1 054.350 joules. Each Btu is defined in terms of the corresponding calorie by 1 Btu/lbm·R ≡ 1 cal/g·K.

From Bolz, R.E. and Tuve, G.L., Units and conversion factors, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, pp. 808–816.

Fundamental Physical Constants

B.N. Taylor, W.H. Parker, and D.N. Langenberg

The numbers in parentheses are the standard deviation uncertainties in the last digits of the quoted value, computed on the basis of internal consistency.

				Units	
Quantity	Symbol	Value	Error, ppm	SI	cgs
Velocity of light	c	2.997250(10)	0.33	10^{8} m sec^{-1}	10^{10} cm sec^{-1}
Fine-structure constant, $[\mu_0 c^2/4\pi](e^2/hc)$	α	7.297351(11)	1.5	10^{-3}	10^{-3}
	α^{-1}	137.03602(21)	1.5		
Electron charge	e	1.6021917(70)	4.4	10^{-19}C	10^{-20} emu
		4.803250(21)	4.4		10^{-10} esu
Planck's constant	h	6.626196(50)	7.6	10^{-34} J·sec	10^{-27} erg·sec
	$h = h/2\pi$	1.0545919(80)	7.6	10^{-34} J·sec	10^{-27} erg·sec
Avogadro's number	N	6.022169(40)	6.6	10^{16} $kmole^{-1}$	10^{23} $mole^{-1}$
Atomic mass unit	amu	1.660531(11)	6.6	10^{-27} kg	10^{-24} g
Electron rest mass	m_e	9.109558(54)	6.0	10^{-31} kg	10^{-28} g
	$m_e{}^{*}$	5.485930(34)	6.2	10^{-4} amu	10^{-4} amu
Proton rest mass	M_p	1.672614(11)	6.6	10^{-27} kg	10^{-24} g
	$M_p{}^{*}$	1.00727661(8)	0.08	amu	amu
Neutron rest mass	M_n	1.674920(11)	6.6	10^{-27} kg	10^{-24} g
	$M_n{}^{*}$	1.00866520(10)	0.10	amu	amu
Ratio of proton mass to electron mass	M_p/m_e	1836.109(11)	6.2		
Electron charge to mass ratio	e/m_e	1.7588028(54)	3.1	10^{11} C Kg^{-1}	10^{7} emu g^{-1}
		5.272759(16)	3.1		10^{17} esu g^{-1}
Magnetic flux quantum, $[c]^{-1}(hc/2e)$	Φ_o	2.0678538(69)	3.3	10^{-15} T·m^2	10^{-7} G·cm^2
	h/e	4.135708(14)	3.3	10^{-15} J·sec C^{-1}	10^{-7} erg·sec emu^{-1}
		1.3795234(46)	3.3		10^{-17} erg·sec esu^{-1}
Quantum of circulation	$h/2m_e$	3.636947(11)	3.1	10^{-4} J·sec kg^{-1}	erg sec g^{-1}
	h/m_e	7.273894(22)	3.1	10^{-4} J·sec kg^{-1}	erg sec g^{-1}
Faraday constant, Ne	F	9.648670(54)	5.5	10^{7} C $kmole^{-1}$	10^{3} emu $mole^{-1}$
		2.892599(16)	5.5		10^{14} esu $mole^{-1}$
Rydberg constant, $[\mu_0 c^2/4\pi]^2(m_e e^4/4\pi h^3 c)$	R_∞	1.09737312(11)	0.10	10^{7} m^{-1}	10^{5} cm^{-1}
Bohr radius, $[\mu_0 c^2/4\pi]^{-1}(h^2/m_e e^2) = \alpha/4\pi R_\infty$	a_0	5.2917715(81)	1.5	10^{-11} m	10^{-9} cm
Classical electron radius, $[\mu_0 c^2/4\pi](e^2/m_e c^2) = \alpha^3/4\pi R_\infty$	r_0	2.817939(13)	4.6	10^{-15} m	10^{-13} cm
Electron magnetic moment in Bohr magnetons	μ_e/μ_s	1.0011596389(31)	0.0031		
Bohr magneton, $[c](eh/2m_e c)$	μ_B	9.274096(65)	7.0	10^{-24} J T^{-1}	10^{-21} erg G^{-1}
Electron magnetic moment	μ_e	9.284851(65)	7.0	10^{-24} J T^{-1}	10^{-21} erg G^{-1}
Gyromagnetic ratio of proton in H_2O	γ_p	2.65751270(82)	3.1	10^{8} rad sec^{-1} T^{-1}	10^{4} rad sec^{-1} FG^{-1}
	$\gamma_p/2\pi$	4.257597(13)	3.1	10^{7} Hz T^{-1}	10^{3} Hz G^{-1}
γ_p corrected for diamagnetism of H_2O	γ_p	2.6751965(82)	3.1	10^{8} rad sec^{-1} T^{-1}	10^{4} rad sec^{-1} G^{-1}
	$\gamma_p/2\pi$	4.257707(13)	3.1	10^{7} Hz T^{-1}	10^{3} Hz G^{-1}
Magnetic moment of protons in H_2O in Bohr magnetons	μ_p/μ_s	1.52099312(10)	0.066	10^{-3}	10^{-3}
Proton magnetic moment in Bohr magnetons	μ_p/μ_s	1.52103264(46)	0.30	10^{-3}	10^{-3}
Proton magnetic moment	μ_p	1.4106203(99)	7.0	10^{-26} J T^{-1}	10^{-23} erg G^{-1}
Magnetic moment of protons in H_2O in nuclear magnetons	μ_p/μ_n	2.792709(17)	6.2		
μ_p/μ_n corrected for diamagnetism of H_2O	μ_p/μ_n	2.792782(17)	6.2		
Nuclear magneton, $[c](eh/2M_p c)$	μ_n	5.050951(50)	10	10^{-27} J T^{-1}	10^{-24} erg G^{-1}
Compton wavelength of the electron, $h/m_e c$	λ_c	2.4263096(74)	3.1	10^{-12} m	10^{-10} cm
	$\lambda_c/2\pi$	3.861592(12)	3.1	10^{-12} m	10^{-10} cm
Compton wavelength of the proton, $h/M_p c$	$\lambda_{c,p}$	1.3214409(90)	6.8	10^{-15} m	10^{-13} cm
	$\lambda_{c,p}/2\pi$	2.103139(14)	6.8	10^{-16} m	10^{-14} cm
Compton wavelength of the neutron, $h/M_n c$	$\lambda_{c,n}$	1.3196217(90)	6.8	10^{-15} m	10^{-13} cm
	$\lambda_{c,n}/2\pi$	2.100243(14)	6.8	10^{-16} m	10^{-14} cm
Gas constant	R_0	8.321434(35)	42	10^{3} J $kmole^{-1}$·K^{-1}	10^{7} erg $mole^{-1}$·K^{-1}
Boltzmann's constant, R_0/N	k	1.380622(59)	43	10^{-23} J K^{-1}	10^{-16} erg K^{-1}
Stefan-Boltzmann constant, $\pi^2 k^4/60h^3c^2$	σ	5.66961(96)	170	10^{-8} W m^{-2} K^4	10^{-5} erg sec^{-1} cm^{-2}·K^{-4}
First radiation constant, $8\pi hc$	c_1	4.992579(38)	7.6	10^{-24} J·m	10^{-15} erg·m

Fundamental Physical Constants (continued)

Quantity	Symbol	Value	Error, ppm	Units SI	Units cgs
Second radiation constant, *hc/k*	c_2	1.438833(61)	43	10^{-2} m·K	cm·K
Gravitational constant	G	6.6732(31)	460	10^{-11} N·m^2 Kg^{-2}	10^{-8} dyn·cm^{2}g^{-2}
kx-unit-to-angstrom conversion factor, $\Lambda = \lambda(\text{Å})/\lambda(\text{kxu})$; $\lambda(\text{Cu}K\alpha_1)$ = 1.537400 kxu	Λ	1.0020764(53)	5.3		
Å*-to-angstrom conversion factor, $\Lambda = \lambda(\text{Å})/\lambda(\text{Å*})$; $\lambda(\text{W}K\alpha_1)$ = 0.2090100 Å*	Λ*	1.0000197(56)	5.6		

* Note that the unified atomic mass scale $^{12}C \equiv 12$ has been used throughout, that amu = atomic mass unit, C = coulomb, G = gauss, Hz = hertz = cycles/sec, J = joule, K = kelvin (degrees kelvin), T = tesla (10^4 G), V = volt, and W = watt. In cases where formulas for constants are given (e.g., R_∞), the relations are written as the product of two factors. The second factor, in parentheses, is the expression to be used when all quantities are expressed in cgs units, with the electron charge in electrostatic units. The first factor, in brackets, is to be included only if all quantities are expressed in SI units. We remind the reader that with the exception of the auxiliary constants which have been taken to the exact, the uncertainties of these constants are correlated, and therefore the general law of error propagation must be used in calculating additional quantities requiring two or more of these constants.

ENERGY CONVERSION FACTORS

Quantity	Value	Unit	Error, ppm
1 kg	5.609538(24)	10^{29} MeV	4.4
1 amu	931.4812(52)	MeV	5.5
Electron mass	0.5110041(16)	MeV	3.1
Proton mass	938.2592(52)	MeV	5.5
Neutron mass	939.5527(52)	MeV	5.5
1 electron volt	1.6021917(70)	10^{-19} J	4.4
		10^{-12} erg	
	2.4179659(81)	10^{14} Hz	3.3
	8.065465(27)	10^5 m^{-1}	3.3
		10^3 cm^{-1}	
	1.160485(49)	10^4 K	42
Energy-wavelength conversion	1.239854(41)	10^{-6} eV·m	3.3
		10^{-4} eV·cm	
Rydberg constant, R_∞	2.179914(17)	10^{-18} J	7.6
		10^{-11} erg	
	13.605826(45)	eV	3.3
	3.2898423(11)	10^{15} Hz	0.35
	1.578936(67)	10^5 K	43
Bohr magneton, μ_s	5.788381(18)	10^{-5} eV T^{-1}	3.1
	1.3996108(43)	10^{10} Hz T^{-1}	3.1
	46.68598(14)	m^{-1}·T^{-1}	3.1
		10^{-1} cm^{-1}·T^{-1}	
	0.671733(29)	K T^{-1}	43
Nuclear magneton, μ_n	3.152526(21)	10^{-8} eV T^{-1}	6.8
	7.622700(42)	10^6 Hz T^{-1}	5.5
	2.542659(14)	10^{-2} m^{-1}·T^{-1}	5.5
		10^{-4} cm^{-1}·T^{-1}	
	3.65846(16)	10^{-4} K T^{-1}	44
Gas constant, R_0	8.20562(35)	10^{-2} m^3·atm kmole^{-1}·K^{-1}	42
Standard volume of ideal gas, V_0	22.4136	m^3 kmole^{-1}	

From Bolz, R.E. and Tuve, G.L., Units and conversion factors, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, pp. 817–818. Originally from *Rev. Mod. Phys.*, 41: 375, 1969. Reprinted by permission of the publisher, American Institute of Physics.

Numerical Constants

π Constants

$\pi =$	3.14159	26535	89793	23846	26433	83279	50288	41971	69399	37510
$1/\pi =$	0.31830	98861	83790	67153	77675	26745	02872	40689	19291	48091
$\pi^2 =$	9.86960	44010	89358	61883	44909	99876	15113	53136	99407	24079
$\log_e \pi =$	1.14472	98858	49400	17414	34273	51353	05871	16472	94812	91531
$\log_{10} \pi =$	0.49714	98726	94133	85435	12682	88290	89887	36516	78324	38044
$\log_{10} \sqrt{2\pi} =$	0.39908	99341	79057	52478	25035	91507	69595	02099	34102	92127

Logarithmic Constants

$e =$	2. 71828	18284	59045	23536	02874	71352	66249	77572	47093	69995
$1/e =$	0. 36787	94411	71442	32159	55237	70161	46086	74458	11131	03176
$e^2 =$	7. 38905	60989	30650	22723	04274	60575	00781	31803	15570	55184
$M = \log_{10} e =$	0. 43429	44819	03251	82765	11289	18916	60508	22943	97005	80366
$1/M = \log_e 10 =$	2. 30258	50929	94045	68401	79914	54684	36420	76011	01488	62877
$\log_{10} M =$	9. 63778	43113	00536	78912	29674	98565	—	10		

Miscellaneous π and *e* Constants

$\pi^4 =$	22. 45915	77183	61045	47342	71522
$e^\pi =$	23. 14069	26327	79269	00572	90864
$e^{-\pi} =$	0. 04321	39182	63772	24977	44177
$e^{1/2\pi} =$	4. 81047	73809	65351	65547	30357
$i^i = e^{-1/2\pi} =$	0. 20787	95763	50761	90854	69556

Numerical Constants

$\sqrt{2} =$	1.41421	35623	73095	04880	16887	24209	69807	85696	71875	37694
$\sqrt[3]{2} =$	1.25992	10498	94873	16476	72106	07278	22835	05702	51464	70150
$\log_e 2 =$	0.69314	71805	59945	30941	72321	21458	17656	80755	00134	36025
$\log_{10} 2 =$	0.30102	99956	63981	19251	37388	94724	49302	67681	89881	46210
$\sqrt{3} =$	1.73205	08075	68877	29352	74463	41505	87326	69428	05253	81038
$\sqrt[3]{3} =$	1.44224	95703	07408	38232	16383	10780	10958	83918	69253	49935
$\log_e 3 =$	1.09861	22886	08109	69139	52452	36922	52570	46474	90557	82274
$\log_{10} 3 =$	0.47712	12547	19662	43729	50279	03255	11530	92001	28864	19069

Miscellaneous

Euler's Constant $\gamma =$	0. 57721	56649	01532	86061					
$\log_e \gamma =$	–0. 54953	93129	81644	82234					
Golden Ratio $\phi =$	1. 61803	39887	49894	84820	45868	34365	63811	77203	09180

Numerical Constants (continued)

Numbers Containing π

$\pi = 3.14159\ 26536$ $\log_{10} \pi = 0.49714\ 98727$ $\log_e \pi = 1.14472\ 98858$

	Number	Logarithm		Number	Logarithm
π	3.1415 927	0.4971 499	π^2	9.8696 044	0.9942 997
2π	6.2831 853	0.7981 799	$2\pi^2$	19.7392 088	1.2953 297
3π	9.4247 780	0.9742 711	$4\pi^2$	39.4784 176	1.5963 597
4π	12.5663 706	1.0092 099	$1/\pi^2$	0.1013 212	9.0057 003 – 10
8π	25.1327 412	1.4002 399	$1/(2\pi^2)$	0.0506 606	8.7046 703 – 10
$\pi/2$	1.5707 963	0.1961 199	$1/(4\pi^2)$	0.0253 303	8.4036 403 – 10
$\pi/3$	1.0471 976	0.0200 286	$\sqrt{\pi}$	1.7724 539	0.2485 749
$\pi/4$	0.7853 982	9.8950 899 – 10	$\sqrt{\pi/4}$ or	0.8862 269	9.9475 449 – 10
$\pi/6$	0.5235 988	9.7189 986 – 10	$\sqrt{\pi}/2$		
$\pi/8$	0.3926 991	9.5940 599 – 10	$\sqrt{\pi}/4$	0.4431 135	9.6465 149 – 10
$2\pi/3$	2.0943 951	0.3210 586	$\sqrt{\pi/2}$	1.2533 141	0.0980 599

	Number	Logarithm		Number	Logarithm
$4\pi/3$	4.1887 902	0.6220 886	$\sqrt{2/\pi}$	0.7978 846	9.9019 401 – 10
$1/\pi$	0.3183 099	9.5028 501 – 10	π^3	31.0062 767	1.4914 496
$2/\pi$	0.6366 198	9.8038 801 – 10	$\sqrt[3]{\pi}$	1.4645 919	0.1657 166
$4/\pi$	1.2732 395	0.1049 101	$1/\sqrt[3]{\pi}$	0.6827 841	9.8342 834 – 10
$1/(2\pi)$	0.1591 549	9.2018 201 – 10	$\sqrt[3]{\pi^2}$	2.1450 294	0.3314 332
$1/(4\pi)$	0.0795 775	8.9007 901 – 10	$1/\sqrt{\pi}$	0.5641 896	9.7514 251 – 10
$1/(6\pi)$	0.0530 516	8.7246 989 – 10	$1/\sqrt{2\pi}$	0.3989 423	9.6009 101 – 10
$1/(8\pi)$	0.0397 887	8.5997 601 – 10	$2/\sqrt{\pi}$	1.1283 792	0.0524 551
$\pi/180$	0.0174 553	8.2418 774 – 10			
$180/\pi$	57.2957 795	1.7581 226			

Change of Base

$$\log_a x = \log_b x / \log_b a$$

$$\log_{10} x = \log_e x / \log_e 10 \qquad \log_e x = \log_{10} x / \log_{10} e$$

$$\log_e x = 1/M \log_{10} x = 2.30258\ 50930 \log_{10} x$$

$$\log_{10} x = M \log_e x = 0.43429\ 44819 \log_e x$$

From Bolz, R.E. and Tuve, G.L., Mathematical and statistical tables, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, p. 877.

Mathematical Constants

Constant	Decimal (Base 10)				Octgal (Base 8)					
					For Use on a Digital Computer					
π Constants										
π	3.14159	26355	89793	23846	3.1103	7552	4210	2643	0215	1423
π^{-1}	0.31830	98861	83790	67153	0.2427	6301	5562	3442	0251	2376
$\sqrt{\pi}$	1.77245	38509	05516	02729	1.6133	7611	0664	7366	5247	4703
π^2	9.86960	44010	89358	61883	11.6751	7144	6762	1357	1322	2556
$\sqrt{2\pi}$	2.5062	82746	31000	50241	2.4033	1143	7754	2340	5454	5371
$(\pi/2)^{1/2}$	1.25331	41373	15500	25120	1.2015	4461	7766	1160	2626	2574
$\pi^{-1/2}$	0.56418	95835	47756	28694	0.4406	7272	4041	2333	3210	6561
$(2\pi)^{-1/2}$	0.39894	22804	01432	67793	0.3142	0424	6365	0331	2043	2077
$\pi^{1/2}$	1.46459	18875	61523	26302	1.3556	7576	3461	0113	3612	7621
$\log_{10} \pi$	0.49714	98726	94133	85435	0.3764	2466	6306	7216	7300	1457
$\ln \pi$	1.14472	98858	49400	17414	1.1120	6404	4347	5033	6413	6537
πe	8.53973	42226	73567		10.4242	6005	5056	5072		
π/e	1.15572	73497	90921		1.1175	6677	3047	0733		
π^e	22.45915	77183	61045	47342	26.3530	5534	1601	0421	1613	1026
e Constants										
e	2.71828	18284	59045	23536	2.5576	0521	3050	5355	1246	5277
e^{-1}	0.36787	94411	71442	32159	0.2742	6530	6613	1674	6761	5272
e^{π}	23.14069	26237	79269	00572	27.1100	2156	5411	1471	4754	6647
$e^{-\pi}$	0.4321	39182	63772	24977	0.0261	0021	1732	6307	3706	4257
$e^{\pi/2}$	4.81407	73809	65351	65547	4.6367	5562	0526	2327	6476	2132
$\log_{10} e$	0.43429	44819	03251	82765	0.3362	6754	2511	5624	1614	5232
Numerical Constants										
$\sqrt{2}$	1.41421	35623	73095	04880	1.3240	4746	3177	1674	6220	4262
$\sqrt[3]{2}$	1.25992	10498	94873	16477						
$\sqrt{3}$	1.73205	08075	68877	29641	1.5666	3656	4130	2312	5167	0145
$\sqrt[3]{3}$	1.44224	95703	07408	38232						
$\log_{10} 2$	0.30102	99956	63981	19251						
$\ln 2$	0.69314	71805	59945	30941	0.5427	1027	7575	0717	3632	5711
$\log_{10} 3$	0.47712	12547	19662	43729						
$\ln 3$	1.09861	22886	68109	69139						
$\ln 10$	2.30258	50929	94045	68401	2.2327	3067	3552	5242	5405	5651
$\log_2 10$					3.24464	741136				
Euler's Constant: γ										
γ	0.57721	56649	01532	86060	0.4474	2147	7067	6660	6172	2321
e^{γ}	1.78107	24179	90197	98522	1.6177	2134	5261	1526	5761	
$e^{-\gamma}$	0.56145	94835	66885	16903	0.4373	5717	0177	1345	7454	
$\log^{\gamma}$	–0.23866	18912	16832	38945	–0.1721	4362	0631	1753	0063	
$\ln \gamma$	–0.54953	93129	81644	82234	–0.4312	7233	6021	7532	2777	

From Bolz, R.E. and Tuve, G.L., Mathematical and statistical tables, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, p. 878.

Derivatives

In the following formulas u, v, w represent functions of x, while a, c, n represent fixed real numbers. All arguments in the trigonometric functions are measured in radians, and all inverse trigonmetric and hyperbolic functions represent principal values.†

1. $\frac{d}{dx}(a)=0$

2. $\frac{d}{dx}(x)=1$

3. $\frac{d}{dx}(au)=a\frac{du}{dx}$

4. $\frac{d}{dx}(u+v-w)=\frac{du}{dx}+\frac{dv}{dx}-\frac{dw}{dx}$

5. $\frac{d}{dx}(uv)=d\frac{dv}{dx}+v\frac{du}{dx}$

6. $\frac{d}{dx}(uvw)=uv\frac{dw}{dx}+vw\frac{du}{dx}+uw\frac{dv}{dx}$

7. $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}=\frac{1}{v}\frac{du}{dx}-\frac{u}{v^2}\frac{dv}{dx}$

8. $\frac{d}{dx}(u^n)=nu^{n-1}\frac{du}{dx}$

9. $\frac{d}{dx}\left(\sqrt{u}\right)=\frac{1}{2\sqrt{u}}\frac{du}{dx}$

10. $\frac{d}{dx}\left(\frac{1}{u}\right)=-\frac{1}{u^2}\frac{du}{dx}$

11. $\frac{d}{dx}\left(\frac{1}{u^n}\right)=-\frac{n}{u^{n+1}}\frac{du}{dx}$

12. $\frac{d}{dx}\left(\frac{u^n}{v^m}\right)=\frac{u^{n-1}}{v^{m+1}}\left(nv\frac{du}{dx}-mu\frac{dv}{dx}\right)$

13. $\frac{d}{dx}(u^n v^m)=u^{n-1}v^{m-1}\left(nv\frac{du}{dx}+mu\frac{dv}{dx}\right)$

14. $\frac{d}{dx}[f(u)]=\frac{d}{du}[f(u)]\cdot\frac{du}{dx}$

† Let $y = f(x)$ and $\frac{dy}{dx} = \frac{d[f(x)]}{dx} = f'(x)$ define respectively a function and its derivative for any value x in their common domain. The differential for the function at such a value x is accordingly defined as

$$dy=d[f(x)]=\frac{dy}{dx}dx=\frac{d[f(x)]}{dx}dx=f'(x)dx$$

Each derivative formula has an associated differential formula. For example, formula 6 above has the differential formula

$$d(uvw)=uv\,dw+vw\,du+uw\,dv$$

15. $\frac{d^2}{dx^2}[f(u)]=\frac{df(u)}{du}\cdot\frac{d^2u}{dx^2}+\frac{d^2f(u)}{du^2}\cdot\left(\frac{du}{dx}\right)^2$

16. $\frac{d^n}{dx^n}[uv]=\binom{n}{0}v\frac{d^nu}{dx^n}+\binom{n}{1}\frac{dv}{dx}\frac{d^{n-1}u}{dx^{n-1}}+\binom{n}{2}\frac{d^2v}{dx^2}\frac{d^{n-2}u}{dx^{n-2}}+\cdots+\binom{n}{k}\frac{d^kv}{dx^k}\frac{d^{n-k}}{dx^{n-k}}+\cdots+\binom{n}{n}u\frac{d^nv}{dx^n}$

    where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ the binomial coefficient, n non-negative integer and $\binom{n}{0} = 1$.

Derivatives (continued)

17. $\frac{du}{dx} = \frac{1}{\frac{dx}{du}} \quad \text{if } \frac{dx}{du} \neq 0$

18. $\frac{d}{dx}(\log_a u) = (\log_a e)\frac{1}{u}\frac{du}{dx}$

19. $\frac{d}{dx}(\log_e u) = \frac{1}{u}\frac{du}{dx}$

20. $\frac{d}{dx}(a^u) = a^u(\log_e a)\frac{du}{dx}$

21. $\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$

22. $\frac{d}{dx}(u^v) = vu^{v-1}\frac{du}{dx} + (\log_e u)u^v\frac{dv}{dx}$

23. $\frac{d}{dx}(\sin u) = \frac{du}{dx}(\cos u)$

24. $\frac{d}{dx}(\cos u) = \frac{du}{dx}(\sin u)$

25. $\frac{d}{dx}(\tan u) = \frac{du}{dx}(\sec^2 u)$

26. $\frac{d}{dx}(\cot u) = -\frac{du}{dx}(\csc^2 u)$

27. $\frac{d}{dx}(\sec u) = \frac{du}{dx}\sec u \cdot \tan u$

28. $\frac{d}{dx}(\csc u) = -\frac{du}{dx}\csc u \cdot \cot u$

29. $\frac{d}{dx}(vers\ u) = \frac{du}{dx}\sin u$

30. $\frac{d}{dx}(arc \sin u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad \left(-\frac{\pi}{2} \le arc \sin u \le \frac{\pi}{2}\right)$

31. $\frac{d}{dx}(arc \cos u) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad (0 \le arc \cos u \le \pi)$

32. $\frac{d}{dx}(arc \tan u) = \frac{1}{1+u^2}\frac{du}{dx}, \quad \left(-\frac{\pi}{2} < arc \tan u \le \frac{\pi}{2}\right)$

33. $\frac{d}{dx}(arc \cot u) = -\frac{1}{1+u^2}\frac{du}{dx}, \quad (0 \le arc \cot u \le \pi)$

34. $\frac{d}{dx}(arc \sec u) = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}, \quad \left(0 \le arc \sec u < \frac{\pi}{2}, -\pi \le arc \sec u < -\frac{\pi}{2}\right)$

35. $\frac{d}{dx}(arc \csc u) = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}, \quad \left(0 < arc \csc u \le \frac{\pi}{2}, -\pi < arc \csc u \le -\frac{\pi}{2}\right)$

36. $\frac{d}{dx}(arc\ vers\ u) = \frac{1}{\sqrt{2u-u^2}}\frac{du}{dx}, \quad (0 \le arc\ vers\ u \le \pi)$

37. $\frac{d}{dx}(\sinh u) = \frac{du}{dx}(\cosh u)$

38. $\frac{d}{dx}(\cosh u) = \frac{du}{dx}(\sinh u)$

Derivatives (continued)

39. $\frac{d}{dx}(\tanh u) = \frac{du}{dx}(\operatorname{sec}h^2 u)$

40. $\frac{d}{dx}(\coth u) = \frac{du}{dx}(\operatorname{csc}h^2 u)$

41. $\frac{d}{dx}(\operatorname{sec}h\, u) = -\frac{du}{dx}(\operatorname{sec}h\, u \cdot \tanh u)$

42. $\frac{d}{dx}(\operatorname{csc}h\, u) = -\frac{du}{dx}(\operatorname{csc}h\, u \cdot \coth u)$

43. $\frac{d}{dx}(\sinh^{-1} u) = \frac{d}{dx}\left[\log\left(u + \sqrt{u^2+1}\right)\right] = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$

44. $\frac{d}{dx}(\cosh^{-1} u) = \frac{d}{dx}\left[\log\left(u + \sqrt{u^2-1}\right)\right] = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad (u>1, \cosh^{-1} u > 0)$

45. $\frac{d}{dx}(\tanh^{-1} u) = \frac{d}{dx}\left[\frac{1}{2}\log\frac{1+u}{1-u}\right] = \frac{1}{1-u^2}\frac{du}{dx}, \quad (u^2<1)$

46. $\frac{d}{dx}(\coth^{-1} u) = \frac{d}{dx}\left[\frac{1}{2}\log\frac{u+1}{u-1}\right] = \frac{1}{1-u^2}\frac{du}{dx}, \quad (u^2>1)$

47. $\frac{d}{dx}(\operatorname{sec}h^{-1} u) = \frac{d}{dx}\left[\log\frac{1+\sqrt{1-u^2}}{u}\right] = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad (0<u<1, \operatorname{sec}h^{-1} u > 0)$

48. $\frac{d}{dx}(\operatorname{csc}h^{-1} u) = \frac{d}{dx}\left[\log\frac{1+\sqrt{1+u^2}}{u}\right] = -\frac{1}{|u|\sqrt{1-u^2}}\frac{du}{dx}$

49. $\frac{d}{dq}\int_p^q f(x)dx = f(q), \quad [p\ constant]$

50. $\frac{d}{dp}\int_p^q f(x)dx = -f(p), \quad [q\ constant]$

51. $\frac{d}{da}\int_p^q f(x,a)dx = \int_p^q \frac{\partial}{\partial a}[f(x,a)]dx + f(q,a)\frac{dq}{da} - f(p,a)\frac{dp}{da}$

From Bolz, R.E. and Tuve, G.L., Mathematical and statistical tables, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, pp. 884–887.

Facts from Algebra

Factors and Expansions

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$
$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$$
$$(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4\,ab^3 + b^4$$
$$a^2 - b^2 = (a - b)\,(a + b)$$
$$a^2 + b^2 = (a + b\sqrt{-1})\,(a - b\sqrt{-1})$$
$$a^3 - b^3 = (a - b)\,(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a + b)\,(a^2 - ab + b^2)$$
$$a^4 + b^4 = (a^2 + ab\sqrt{2} + b^2)\,(a^2 - ab\sqrt{2} + b^2)$$
$$a^n - b^n = (a - b)\,(a^{n-1} + a^{n-2}b + \dots + b^{n-1})$$
$$a^n - b^n = (a + b)\,(a^{n-1} - a^{n-2}b + \dots - b^{n-1}), \text{ for even values of } n$$
$$a^n + b^n = (a + b)\,(a^{n-1} - a^{n-2}b + \dots + b^{n-1}), \text{ for odd values of } n$$
$$a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)\,(a^2 - ab + b^2)$$
$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$
$$(a + b + c)^3 = a^3 + b^3 + c^3 + 3a^2(b + c) + 3b^2(a + c) + 3c^2(a + b) + 6abc$$
$$(a + b + c + d + \dots)^2 = a^2 + b^2 + c^2 + d^2 + \dots$$
$$+\, 2a(b + c + d + \dots) + 2b(c + d + \dots) + 2c(d + \dots) + \dots$$

Powers and Roots

$$a^x \times a^y = a^{(x+y)} \qquad a^0 = 1 \text{ [if } a - 0] \qquad (ab)^x = a^x b^x$$

$$\frac{a^x}{a^y} = a^{(x-y)} \qquad a^{-x} = \frac{1}{a^x} \qquad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$$\left(a^x\right)^y = a^{xy} \qquad a^{\frac{1}{x}} = \sqrt[x]{a} \qquad \sqrt[x]{ab} = \sqrt[x]{a}\sqrt[x]{b}$$

$$\sqrt[x]{\sqrt[y]{a}} = \sqrt[xy]{a} \qquad a^{\frac{x}{y}} = \sqrt[y]{a^x} \qquad \sqrt[x]{\frac{a}{b}} = \frac{\sqrt[x]{a}}{\sqrt[x]{b}}$$

Proportion

$$\textit{If}\quad \frac{a}{b} = \frac{c}{d}, \qquad \text{then} \qquad \frac{a+b}{b} = \frac{c+d}{d}\;\frac{a-b}{b} = \frac{c-d}{d}, \qquad \frac{a-b}{a+b} = \frac{c-d}{c+d}$$

From Bolz, R.E. and Tuve, G.L., Mathematical and statistical tables, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, p. 887.

Integrals—Elementary Forms

1. $\int a\,dx = ax$

2. $\int a \cdot f(x)\,dx = a\int f(x)\,dx$

3. $\int \phi(y)\,dx = \int \frac{\phi(y)}{y'}\,dy$, where $y' = \frac{dy}{dx}$

4. $\int (u+v)\,dx = \int u\,dx + \int v\,dx$, where u and v are any functions of x

5. $\int u\,dv = u\int dv - \int v\,du = uv - \int v\,du$

6. $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

7. $\int x^n dx = \frac{x^{n+1}}{n+1}$, except $n = -1$

8. $\int \frac{f'(x)dx}{f(x)} = \log f(x)$, $\left(df(x) = f'(x)dx\right)$

9. $\int \frac{dx}{x} = \log x$

10. $\int \frac{f'(x)dx}{2\sqrt{f(x)}} = \sqrt{f(x)}$, $\left(df(x) = f'(x)dx\right)$

11. $\int e^x dx = e^x$

12. $\int e^{ax} dx = e^{ax}/a$

13. $\int b^{ax} dx = \frac{b^{ax}}{a\log b}$, $(b > 0)$

14. $\int \log x\,dx = x\log x - x$

15. $\int a^x \log a\,dx = a^x$, $(a > 0)$

16. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

17. $$\int \frac{dx}{a^2 - x^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\frac{x}{a} \\ \text{or} \\ \frac{1}{2a}\log\frac{a+x}{a-x}, & (a^2 > x^2) \end{cases}$$

18. $$\int \frac{dx}{x^2 - a^2} = \begin{cases} -\frac{1}{a}\coth^{-1}\frac{x}{a} \\ \text{or} \\ \frac{1}{2a}\log\frac{x-a}{x+a}, & (x^2 > a^2) \end{cases}$$

Integrals—Elementary Forms (continued)

19. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \begin{cases} \sin^{-1} \dfrac{x}{|a|} \\ \quad \text{or} \\ -\cos^{-1} \dfrac{x}{|a|}, \quad (a^2 > x^2) \end{cases}$$

20. $$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\left(x + \sqrt{x^2 \pm a^2}\right)$$

21. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|}\sec^{-1}\frac{x}{a}$$

22. $$\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{a^2 \pm x^2}}{x}\right)$$

From Bolz, R.E. and Tuve, G.L., Mathematical and statistical tables, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, pp. 888–889.

Series

The expression in parentheses following certain of the series indicates the region of convergence. If not otherwise indicated it is to be understood that the series converges for all finite values of x.

Binomial

$$(x-y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots (y^2 < x^2)$$

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \frac{n(n-1)(n-2)x^3}{3!} + \cdots etc. \quad (x^2 < 1)$$

$$(1 \pm x)^{-n} = 1 \mp nx + \frac{n(n+1)x^2}{2!} \mp \frac{n(n+1)(n+2)x^3}{3!} + \cdots etc. \quad (x^2 < 1)$$

$$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \cdots \quad (x^2 < 1)$$

$$(1 \pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp 6x^5 + \cdots \quad (x^2 < 1)$$

Reversion of Series

Let a series be represented by

$$y = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots \quad (a_1 \neq 0)$$

to find the coefficients of the series

$$x = A_1y + A_2y^2 + A_3y^3 + A_4y^4 + \cdots$$

$$A_1 = \frac{1}{a_1} \quad A_2 = -\frac{a_2}{a_1^3} \quad A_3 = \frac{1}{a_1^5}(2a_2^2 - a_1a_3)$$

$$A_4 = \frac{1}{a_1^7}(5a_1a_2a_3 - a_1^2a_4 - 5a_2^3)$$

$$A_5 = \frac{1}{a_1^9}(6a_1^2a_2a_4 + 3a_1^2a_3^2 + 14a_2^4 - a_1^3a_5 - 21a_1a_2^2a_3)$$

$$A_6 = \frac{1}{a_1^{11}}(7a_1^3a_2a_5 + 7a_1^3a_3a_4 + 84a_1a_2^3a_3 - a_1^4a_6 - 28a_1^2a_2^2a_4 - 28a_1^2a_2a_3^2 - 42a_2^5)$$

$$A_7 = \frac{1}{a_1^{13}}(8a_1^4a_2a_6 + 8a_1^4a_3a_5 + 4a_1^4a_4^2 + 120a_1^2a_2^3a_4 + 180a_1^2a_2^2a_3^2$$
$$+ 132a_2^6 - a_1^5a_7 - 36a_1^3a_2^2a_5 - 72a_1^3a_2a_3a_4 - 12a_1^3a_3^3 - 330a_1a_2^4a_3)$$

Taylor

1. $$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a)$$
$$+ \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots (\text{Taylor's Series})$$

(Increment form)

2. $$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$
$$= f(h) + xf'(h) + \frac{x^3}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \cdots$$

3. If $f(x)$ is a function possessing derivatives of all orders throughout the interval $a \leqq x \leqq b$, then there is a value X, with $a < X < b$, such that

Series (continued)

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\cdots$$
$$+\frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{(b-a)^n}{n!}f^{(n)}(X)$$

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)$$
$$+\frac{h^n}{n!}f^{(n-1)}(a+\theta h),\quad b=a+h,\quad 0<\theta<1$$

or

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+(x-a)^{n-1}\frac{f^{(n-1)}(a)}{(n-1)!}+R_n$$

where

$$R_n=\frac{f^{(n)}\left[a+\theta\cdot(x-a)\right]}{n!}(x-a)^n,\quad 0<\theta<1$$

The above forms are known as Taylor's series with the remainder term.

4. *Taylor's series for a function of two variables*

If $\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x,y)=h\frac{\partial f(x,y)}{x}+k\frac{\partial f(x,y)}{\partial y}$;

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{y}\right)^2 f(x,y)=h^2\frac{\partial^2 f(x,y)}{\partial x^2}+2hk\frac{\partial^2 f(x,y)}{\partial x\partial y}+k^2\frac{\partial^2 f(x,y)}{\partial y^2}$$

etc., and if $h\left(\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x,y)\Big|_{\substack{x=a\\y=b}}$ with the bar and subscripts means that after differentiation we are to replace x by a and y by b,

$$f(a+h,b+k)=f(a,b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x,y)\Big|_{\substack{x=a\\y=b}}+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x,y)\Big|_{\substack{x=a\\y=b}}+\cdots$$

Maclaurin

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\cdots+x^{n-1}\frac{f^{(n-1)}(0)}{(n-1)!}+R_n$$

where

$$R_n=\frac{x^n f^{(n)}(\theta x)}{n!},\quad 0<\theta<1$$

Exponential

$$e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\quad(\text{all real values of }x)$$

Series (continued)

$$a^x = 1 + x\log_e a + \frac{(x\log_e a)^2}{2!} + \frac{(x\log_e a)^3}{3!} + \cdots$$

$$e^x = e^a\left[1 + (x-a) + \frac{(x-a)^2}{2!} + \frac{(x-a)^3}{3!} + \cdots\right]$$

Logarithmic

$$\log_e x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots \qquad \left(x > \frac{1}{2}\right)$$

$$\log_e x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots \qquad (2 \geq x > 0)$$

$$\log_e x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right] \quad (x > 0)$$

$$\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \qquad (-1 < x < 1)$$

$$\log_e(n+1) - \log_e(n-1) = 2\left[\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \cdots\right]$$

$$\log_e(a+x) = \log_e a + 2\left[\frac{x}{2a+x} + \frac{1}{3}\left(\frac{x}{2a+x}\right)^3 + \frac{1}{5}\left(\frac{x}{2a+x}\right)^5 + \cdots\right] \quad (a > 0, -a < x < +\infty)$$

$$\log_e \frac{1+x}{1-x} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots\right], \quad -1 < x < 1$$

$$\log_e x = \log_e a + \frac{(x-a)}{a} - \frac{(x-a)^2}{2a^2} + \frac{(x-a)^2}{3a^3} - + \cdots, \quad 0 < x \leqq 2a$$

Trigonometric

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (\text{all real values of } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (\text{all real values of } x)$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1} + \cdots,$$

$$\left[x^2 < \frac{\pi^2}{4}, \text{ and } B_n \text{ represents the } n\text{'th Bernoulli number.}\right]$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \frac{x^7}{4725} - \cdots - \frac{2^{2n}B_{2n}}{(2n)!}x^{2n-1} - \cdots,$$

$$\left[x^2 < \pi^2, \text{ and } B_n \text{ represents the } n\text{'th Bernoulli number.}\right]$$

$$\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots + \frac{E_{2n}x^{2n}}{(2n)!} + \cdots,$$

$$\left[x^2 < \frac{\pi^2}{4}, \text{ and } B_n \text{ represents the } n\text{'th Euler number.}\right]$$

Series (continued)

$$\csc x = \frac{1}{x} - \frac{x}{6} + \frac{7}{360}x^3 + \frac{31}{15{,}120}x^5 + \frac{127}{604{,}800}x^7 + \cdots + \frac{2\left(2^{2n-1}-1\right)}{(2n)!}B_{2n}x^{2n-1} + \cdots,$$

$\left[x^2 < \pi^2\right.$, and B_n represents the n'th Bernoulli number.$\left.\right]$

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\cdots \qquad \left(x^2<\infty\right)$$

$$\cos x = \left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right)\left(1-\frac{4x^2}{5^2\pi^2}\right)\cdots \qquad \left(x^2<\infty\right)$$

$$\sin^{-1} x = x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \cdots \qquad \left(x^2<1, -\frac{\pi}{2} < \sin^{-1} x < \frac{\pi}{2}\right)$$

$$\cos^{-1} x = \frac{\pi}{2} - \left(x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3x^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots\right) \qquad \left(x^2<1, 0<\cos^{-1} x<\pi\right)$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \left(x^2<1\right)$$

$$\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^2} - \frac{1}{5x^5} + \frac{1}{7x^7} - \cdots \qquad \left(x>1\right)$$

$$\tan^{-1} x = -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^2} - \frac{1}{5x^5} + \frac{1}{7x^7} - \cdots \qquad \left(x<-1\right)$$

$$\cot^{-1} x = \frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots \qquad \left(x^2<1\right)$$

$$\log_e \sin x = \log_e x - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \cdots \qquad \left(x^2<\pi^2\right)$$

$$\log_e \cos x = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \cdots \qquad \left(x^2<\frac{\pi^2}{4}\right)$$

$$\log_e \tan x = \log_e x + \frac{x^2}{3} + \frac{7x^4}{90} + \frac{62x^6}{2835} + \cdots \qquad \left(x^2<\frac{\pi^2}{4}\right)$$

$$e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} - \frac{3x^6}{6!} + \frac{56x^7}{7!} + \cdots$$

$$e^{\cos x} = e\left(1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \cdots\right)$$

$$e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \frac{37x^5}{5!} + \cdots \qquad \left(x^2<\frac{\pi^2}{4}\right)$$

$$\sin x = \sin a + (x-a)\cos a - \frac{(x-a)^2}{2!}\sin a - \frac{(x-a)^3}{3!}\cos a + \frac{(x-a)^4}{4!}\sin a + \cdots$$

Hyperbolic and Inverse Hyperbolic

Table of expansion of certain functions into power series

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots \qquad |x|<\infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots \qquad |x|<\infty$$

Series (continued)

$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \frac{62}{2835}x^9 - \cdots + \frac{(-1)^{n+1}2^{2n}(2^{2n}-1)}{(2n)!}B_{2n}x^{2n-1} \pm \cdots^{(1)}$	$\lvert x \rvert < \frac{\pi}{2}$
$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \frac{x^7}{4725} + \cdots + \frac{(-1)^{n+1}2^{2n}}{(2n)!}B_{2n}x^{2n-1} \pm \cdots^{(1)}$	$0 < \lvert x \rvert < \pi$
$\mathrm{sech}\, x == 1 - \frac{1}{2!}x^2 + \frac{5}{4!}x^4 - \frac{61}{6!}x^6 + \frac{1385}{8!}x^8 - \cdots + \frac{(-1)^n}{(2n)!}E_{2n}x^{2n} \pm \cdots^{(2)}$	$\lvert x \rvert < \frac{\pi}{2}$
$\mathrm{cosech}\, x = \frac{1}{x} - \frac{x}{6} + \frac{7x^3}{360} - \frac{31x^5}{15{,}120} + \cdots + \frac{2(-1)^n(2^{2n-1}-1)}{(2n)!}B_{2n}x^{2n-1} + \cdots^{(1)}$	$0 < \lvert x \rvert < \pi$
$\mathrm{arg}\sinh x = x - \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \cdots + (-1)^n \cdot \frac{1\cdot 3\cdot 5(2n-1)}{2\cdot 4\cdot 6\ldots 2n(2n+1)}x^{2n+1} \pm \cdots$	$\lvert x \rvert < 1$
$\mathrm{arg}\cosh x = \pm\left[\ln(2x) - \frac{1}{2\cdot 2x^2} - \frac{1\cdot 3}{2\cdot 4\cdot 4x^4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6x^6} - \cdots\right]$	$x > 1$
$\mathrm{arg}\tanh x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots$	$\lvert x \rvert < 1$
$\mathrm{arg}\coth x = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \frac{1}{7x^7} + \cdots + \frac{1}{(2n+1)x^{2n+1}} + \cdots$	$\lvert x \rvert > 1$

(1) B_n denotes Bernoull's numbers.
(2) E_n denotes Euler's numbers

Fourier

1. If $f(x)$ is a bounded periodic function of period $2L$ (i.e., $f(x + 2L) = f(x)$), and satisfies the *Dirichlet conditions*:
   a) In any period $f(x)$ is continuous, except possibly for a finite number of jump discontinuities.
   b) In any period $f(x)$ has only a finie number of maxima and minima.
   Then $f(x)$ may be represented by the *Fourier series*

$$\frac{a_0}{2} + \sum_{n-1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where a_n and b_n are as determined below. This series will converge to $f(x)$ at every point where $f(x)$ is continuous, and to

$$\frac{f(x^+) + f(x^-)}{2}$$

(i.e., the average of the left-hand and right-hand limits) at every point where $f(x)$ has a jump discontinuity.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0,1,2,3,\ldots;$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1,2,3,\ldots$$

Series (continued)

We may also write

$$a_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\cos\frac{n\pi x}{L}dx \quad and \quad b_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\sin\frac{n\pi x}{L}dx$$

where a is any real number. Thus if $\alpha = 0$,

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0,1,2,3,\ldots;$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1,2,3,\ldots$$

2. If in addition to the above restrictions, $f(x)$ is even (i.e., $f(-x) = f(x)$), the Fourier series reduces to

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

That is, $b_n = 0$. In this case, a simpler formula for a_n is

$$a_n = \frac{2}{L}\int_0^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0,1,2,3,\ldots$$

3. If in addition to the restrictions in (1), $f(x)$ is an odd function (i.e., $f(-x) = -f(x)$), then the Fourier series reduces to

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

That is, $a_n = 0$. In this case, a simpler formula for the b_n is

$$b_n = \frac{2}{L}\int_0^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1,2,3,\ldots$$

4. If in addition to the restrictions in (2) above, $f(x) = -f(L - x)$, then a_n will be 0 for all even values of n, including $n = 0$. Thus in this case, the expansion reduces to

$$\sum_{m=1}^{\infty} a_{2m-1} \cos\frac{(2m-1)\pi x}{L}$$

5. If in addition to the restrictions in (3) above, $f(x) = f(L - x)$, then b_n will be 0 for all even values of n. Thus in this case, the expansion reduces to

$$\sum_{m=1}^{\infty} b_{2m-1} \sin\frac{(2m-1)\pi x}{L}$$

(The series in (4) and (5) are known as *odd-harmonic series*, since only the odd hormonics appear. Similar rules may be stated for even-harmonic series, but when a series appears in the even-harmonic form, it means that $2L$ has not been taken as the smallest period of $f(x)$. Since any integral multiple of a period is also a period, series obtained in this way will also work, but in general computation is simplified if $2L$ is taken to be the smallest period.)

6. If we write the Euler definitions for cos θ and sin θ, we obtain the complex form of the Fourier Series known either as the “Complex Fourier Series” or the “Exponential Fourier Series” of $f(x)$. It is represented as

$$f(x) = \frac{1}{2}\sum_{n=-\infty}^{n=+\infty} c_n e^{i\omega_n x}$$

where

$$c_n = \frac{1}{L}\int_{-L}^{L} f(x)e^{-i\omega_n x}dx, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Series (continued)

with $\omega_n = \frac{n\pi}{L}, \quad n = 0, \pm 1, \pm 2, \ldots$

The set of coefficients $\{c_n\}$ is often referred to as the Fourier spectrum.

7. If both sine and cosine terms are present and if f(x) is of period $2L$ and expandable by a Fourier series, it can be represented as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L} + \phi_n\right), \quad where\ a_n = c_n \sin\phi_n,$$

$$b_n = c_n \cos\phi_n, \quad c_n = \sqrt{a_n^2 + b_n^2}, \quad \phi_n = arc\tan\left(\frac{a_n}{b_n}\right)$$

It can also be represented as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L} + \phi_n\right), \quad where\ a_n = c_n \cos\phi_n,$$

$$b_n = -c_n \sin\phi_n, \quad c_n = \sqrt{a_n^2 + b_n^2}, \quad \phi_n = arc\tan\left(-\frac{b_n}{a_n}\right)$$

where ϕ_n is chosen so as to make a_n, b_n, and c_n hold.

8. The following table of trigonometric identities should be helpful for developing Fourier Series.

	n	n even	n odd	$n/2$ odd	$n/2$ even
$\sin n\pi$	–0	0	0	0	0
$\cos n\pi$	$(-1)^n$	+1	–1	+1	+1
$^*\sin\frac{n\pi}{2}$		0	$(-1)^{(n-1)/2}$	0	0
$^*\cos\frac{n\pi}{2}$		$(-1)^{n/2}$	0	–1	+1
$\sin\frac{n\pi}{4}$			$\frac{\sqrt{2}}{2}(-1)^{\left(n^2+4n+11\right)/8}$	$(-1)^{(n-2)/4}$	0

* A useful formula for $\sin\frac{n\pi}{2}$ *and* $\cos\frac{n\pi}{2}$ is given by

$$\sin\frac{n\pi}{2} = \frac{(i)^{n+1}}{2}\left[(1-1)^n - 1\right]\ and\ \cos\frac{n\pi}{2} = \frac{(i)^n}{2}\left[(-1)^n + 1\right],\ where\ i^2 = -1$$

Auxiliary Formulas for Fourier Series

$$1 = \frac{4}{\pi}\left[\sin\frac{\pi x}{k} + \frac{1}{3}\sin\frac{3\pi x}{k} + \frac{1}{5}\sin\frac{5\pi x}{k} + \cdots\right] \qquad [0 < x < k]$$

$$x = \frac{2k}{\pi}\left[\sin\frac{\pi x}{k} - \frac{1}{2}\sin\frac{2\pi x}{k} + \frac{1}{3}\sin\frac{3\pi x}{k} - \cdots\right] \qquad [-k < x < k]$$

$$x = \frac{k}{2} - \frac{4k}{\pi^2}\left[\cos\frac{\pi x}{k} + \frac{1}{3^2}\cos\frac{3\pi x}{k} + \frac{1}{5^2}\cos\frac{5\pi x}{k} + \cdots\right] \qquad [0 < x < k]$$

Series (continued)

$$x^2 = \frac{2k^2}{\pi^3}\left[\left(\frac{\pi^2}{1} - \frac{4}{1}\right)\sin\frac{\pi x}{k} - \frac{\pi^2}{2}\sin\frac{2\pi x}{k} + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin\frac{3\pi x}{k} - \frac{\pi^2}{4}\sin\frac{4\pi x}{k} + \left(\frac{\pi^2}{5} - \frac{4}{5^3}\right)\sin\frac{5\pi x}{k} + \cdots\right] \qquad [0 < x < k]$$

$$x^2 = \frac{k^2}{3} - \frac{4k^2}{\pi^2}\left[\cos\frac{\pi x}{k} - \frac{1}{2^2}\cos\frac{2\pi x}{k} + \frac{1}{3^2}\cos\frac{3\pi x}{k} - \frac{1}{4^2}\cos\frac{4\pi x}{k} + \cdots\right] \qquad [-k < x < k]$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \cdots = \frac{\pi^2}{24}$$

From Bolz, R.E. and Tuve, G.L., Mathematical and statistical tables, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, pp. 890–897.

Tables of Statistical Probability

Mathematical probability deals with the random or chance variation of numerical data. When the probability or statistical chance is expressed numerically (percentage or decimal), it is a specific likelihood representing the ratio of chances in favor to total chances available. In the usual probability-distribution graphs probability is represented by an area under the frequency curve.

Measures of central tendency are the *mean* (μ) or arithmetical average, the *median* or middle value, and the *mode* or most frequent value.

Measures of dispersion are the individual *deviation* (x), which is the difference between the mean and the specific value under consideration; the *standard deviation* (σ), which is the square root of the mean of the squares of the deviations (rms†); and the *variance*, which is the square of the standard deviation (σ^2). The *range* is the spread between the smallest and the largest items of data.

Frequency (y) is a measure of the importance of a given value in terms of the frequency of its occurrence. It is commonly expressed as the frequency of occurrence of stated values of the deviation from y_o, the mean, but it also refers to the frequency of occurrence of a given magnitude in original data.

Frequency distribution describes the frequency of occurrence of the various numerical values, or the frequency of occurrence of stated deviations from the mean. A frequency distribution is represented mathematically, by equations, curves, or tables. The distributions covered by the following tables include he normal, the binomial, the Poisson, the *t*, the *F*, and the chi-square distributions.

Statistical sample is a random sample representative of all the original data. The data being sampled are referred to collectively as the *population* or the *universe*.

Statistical significance is a general term for the assumed importance of the probability. The range of probabilities used to describe a very likely occurrence is often expressed as a percentage between 90 and 99.9. For very unlikely occurrences the probabilities between 0.1 percent and 10 percent are examined. The arbitrarily selected percentages are often called *confidence limits*, implying that there is a 100 percent probability, representing full or complete certainty. The borderline between a "significant" probability and one that is not significant must be arbitrarily selected. (See Student's *t*-Distribution.)

Degrees of freedom (N) refer to the number of independent properties of a sample. It is usually n-1, where n is the number of data items.

Table 1. Normal or Gaussian Probability Distribution.‡ This "continuous" distribution is applicable to a population or universe for which the number of items of data is infinitely large and in which the deviations from the mean are random and unrelated. It applies also to a large representative sample of such data, such as 50 or 100 items; the larger the sample, the closer the approximation. For this symmetrical distribution the mean, the median, and the mode all coincide and are represented by the maximum ordinate y_o. The table gives normalized values, in which the maximum probability area under the curve is unity; the deviations are measured in units of σ, the mean deviation, and the maximum ordinate is $\frac{1}{\sqrt{2\pi}} = 0.39894$.

Table 2. Student's *t*-Distribution. The *t*-test is widely used to evaluate the significance of differences, such as the difference between the means of two samples and the difference between a sample mean and the population mean. At the top of each column in the table is that probability that the difference would exist by chance alone. The probability of a match or a fit decreases as *t* increases. Two common borderline values are $p = 0.01$ and $p = 0.05$. For example, if the computed value of *t* is larger than the one given in the column headed *0.05*, the interpretation might be as follows: the probability that this difference is due to chance alone is less than one in twenty; hence the difference is significant and is due to factors other than pure chance. The ratio *t* must be correctly computed.** For comparing a sample with a known parent population, it is the ratio of the difference between the sample mean and the population mean to the standard deviation of the mean of the parent population (corrected for sample size):

$$t = \frac{mean_1 - mean_2}{\sigma / \sqrt{N}}$$

The *t*-distribution approaches the normal distribution as the number of degrees of freedom approaches infinity, but in any case the means themselves are assumed to be normally distributed.

Table 3. Chi-Square Distribution. This is another test for the significance of differences by evaluation of the spread of the data. There are several ways to apply the chi-square test. One involves the ratio of the squares of the two standard deviations:

$$\text{chi-square} = N(\sigma_1/\sigma_2)^2$$

† For linear correlation by a line of regression, using least squares, the standard deviation of the points from the line is called the "standard error of estimate."

‡ Also called the normal error function and the normal frequency curve.

** Consult a textbook on statistics; see References.

Tables of Statistical Probability (continued)

For example, if it is desired to test the variability or dispersion for a sample when that of a parent population is known, a value of chi-square (= *N* times the ratio of variances) larger than the one in the *0.10* column would mean that there are fewer than ten chances in one hundred that the sample represented the parent population, and that its larger variability occurred purely by chance. As the chi-square increases, the probability of matching or agreement decreases.

Another application of the chi-square table uses the summation of the squares of frequency differences for goodness of fit with the parent distribution:

$$chi\text{-}square = \frac{\Sigma(y - y_c)^2}{y_c}$$

where the values of y_c are those of the comparison standard.

A very useful application of the chi-square test is in the evaluation of attribute data where there are a number of classes and the expectations in the different classes are unequal.

Table 4. ***F*-Distribution.** This distribution is used for testing dispersion in terms of variance. One use for the *F*-test is to determine whether two samples, possibly of different sizes, drawn independently from two normal populations, actually represent populations with identical standard deviations. Here *F* is the ratio of the variance of the *samples*:

$$F = \sigma_1^2 / \sigma_2^2$$

In the tables, since the two sample sizes and degrees of freedom may be unequal, the additional variable is accommodated by setting up a separate table for each probability value, the *p* values used here being in the range 0.001 to 0.10. The borderline between a significant difference and one that is not significant must be selected, and the table with that probability value is used. If the value of *F* is larger than the corresponding one in that table, the probability that the two samples came from like populations is even less than that selected as a borderline.

Table 5. Binomial Distribution. This is a "discrete" distribution representing the probabilities of "success" in *N* trials for a population or sample in which only two outcomes are possible, but for which the eventual outcome is fixed and known if an infinite number of trials are made.† This eventual outcome is fixed by the conditions, such as 0.5 for one face of a coin or 0.1667 for one face of a six-sided die. Values in the body of the table represent the cumulative probability of *X* or more successes in *N* trials. In applications to acceptance or attribute sampling, the table gives the probability of *X* or more acceptances (or rejections) in a single sample of N items. In either case the known or fixed probability of the result (success or failure) for the entire population is represented by *p*.

Table 6. Poisson Distribution. This is a discrete distribution approximating the binomial when the total number of items of data (the populations) is very large, but the probability (*p*) is very small and the sample is small compared with the population

The Poisson cumulative probability, i.e., the probability that *X* is greater than or equal to *X′*, is expressed as

$$P = \sum_{X=X'}^{\infty} \frac{e^{-N_p}(Np)^x}{X'}$$

for specified values of *X′* and *Np*.

The table is arranged in terms of the product, *Np*, where *N* is the sample size and *p* is the fixed probability for the entire population. For this distribution $Np = \mu = \sigma^2$, i.e., both the mean and the variance are equal to *Np*. The standard deviation is $\sigma = \sqrt{Np}$. Values in the body of the table represent the cumulative probability of *X* or more successes in *N* trials (the same as for the binomial table); or in sampling, the values represent the probability of *X* or more acceptances in sample of *N* items. In either case the fixed probability for the whole population is *p*.

†Outcomes might be expressed as successes or failure, yes or no, hit or miss, accept or reject, heads or talk, plus or minus, one or zero.

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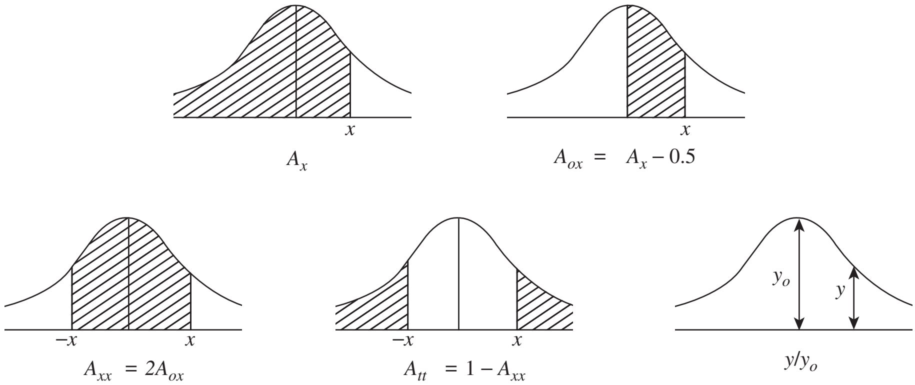
Table 1. Ordinates and Areas for Normal or Gaussian Probability Distribution



SYMBOLS:

x = deviation from the mean (or from zero error). One unit of x equals one standard deviation.

y = frequency of occurrence of the deviation ("probability density")

$$= f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

σ = standard deviation or error (rms) = $\sqrt{\frac{\Sigma x^2}{n}}$

y_o = frequency of occurrence of mean value

y/y_o = relative frequency in terms of mean frequency

A_x = area under curve from $-\alpha$ to x = $\int_{-\infty}^{+x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$

A_{ox} = area under curve from zero to x

A_{xx} = area under curve from $-x$ to $+x$ = probability of occurrence of values deviating from mean value in range from $-x$ to $+x$

A_{tt} = residual area, in the two "tails" = $1 - A_{xx}$

Note: All areas in the table are based on a transformation of the variable such that $A_{-\infty}^{+\infty} = 1$, with decimal values shown in the table.

Independent Variable = Deviation = x/σ

x/σ	A_{xx}	A_{ox}	A_x	A_{tt}	y	y/y_o
.00	.0000	.0000	.5000	1.0000	.3989	1.0000
.05	.0398	.0199	.5199	.9602	.3984	.9986
.10	.0797	.0398	.5398	.9203	.3971	.9950
.15	.1192	.0596	.5595	.8808	.3945	.9888
.20	.1585	.0793	.5793	.8415	.3910	.9802
.25	.1774	.0987	.5987	.8026	.3867	.9692
.30	.2358	.1179	.6179	.7642	.3814	.9560
.35	.2736	.1368	.6368	.7263	.3752	.9405
.40	.3108	.1554	.6554	.6892	.3683	.9231
.45	.3472	.1736	.6736	.6527	.3605	.9037
.50	.3829	.1915	.6915	.6171	.3521	.8825
.55	.4176	.2088	.7088	.5823	.3429	.8596
.60	.4515	.2257	.7257	.5485	.3332	.8353
.65	.4844	.2422	.7422	.5157	.3230	.8096
.70	.5161	.2580	.7580	.4839	.3123	.7827
.75	.5468	.2734	.7734	.4533	.3011	.7548
.80	.5763	.2881	.7881	.4237	.2897	.7262
.85	.6046	.3023	.8023	.3953	.2780	.6968
.90	.6319	.3159	.8159	.3681	.2661	.6670
.95	.6578	.3289	.8289	.3421	.2541	.6368

Table 1. Ordinates and Areas for Normal or Gaussian Probability Distribution (continued)

x/σ	A_{xx}	A_{ox}	A_x	A_{tt}	y	y/y_o
1.00	.6827	.3413	.8413	.3173	.2420	.6065
1.05	.7062	.3531	.8531	.2938	.2299	.5762
1.10	.7286	.3643	.8643	.2714	.2179	.5461
1.15	.7498	.3749	.8749	.2502	.2059	.5162
1.20	.7698	.3849	.8849	.2302	.1942	.4868
1.25	.7887	.3944	.8944	.2113	.1826	.4578
1.30	.8064	.4032	.9032	.1936	.1714	.4296
1.35	.8229	.4115	.9115	.1771	.1604	.4020
1.40	.8384	.4192	.9192	.1616	.1497	.3753
1.45	.8530	.4265	.9265	.1470	.1394	.3495
1.50	.8664	.4332	.9332	.1336	.1295	.3247
1.55	.8788	.4394	.9394	.1212	.1200	.3008
1.60	.8904	.4452	.9452	.1096	.1109	.2780
1.65	.9010	.4505	.9505	.0990	.1023	.2563
1.70	.9108	.4554	.9554	.0892	.0940	.2376
1.75	.9198	.4599	.9599	.0802	.0863	.2163
1.80	.9281	.4641	.9641	.0720	.0790	.1979
1.85	.9356	.4678	.9678	.0644	.0721	.1806
1.90	.9426	.4713	.9713	.0574	.0656	.1645
1.95	.9488	.4744	.9744	.0512	.0596	.1494
2.00	.9545	.4772	.9772	.0455	.0540	.1353
2.05	.9596	.4798	.9798	.0404	.0488	.1223
2.10	.9642	.4821	.9821	.0358	.0440	.1040
2.15	.9684	.4842	.9842	.0316	.0396	.0992
2.20	.9722	.4861	.9861	.0278	.0355	.0890
2.25	.9756	.4878	.9878	.0244	.0317	.0796
2.30	.9786	.4893	.9893	.0214	.0283	.0709
2.35	.9812	.4906	.9906	.0188	.0252	.0632
2.40	.9836	.4918	.9918	.0164	.0224	.0561
2.45	.9858	.4929	.9929	.0143	.0198	.0497
2.50	.9876	.4938	.9938	.0124	.0175	.0439
2.55	.9892	.4946	.9946	.0108	.0155	.0387
2.60	.9907	.4953	.9953	.0093	.0136	.0341
2.65	.9920	.4960	.9960	.0080	.0119	.0299
2.70	.9930	.4965	.9965	.0070	.0104	.0261
2.75	.9940	.4970	.9970	.0060	.0091	.0228
2.80	.9948	.4974	.9974	.0051	.0079	.0198
2.85	.9956	.4978	.9978	.0044	.0069	.0172
2.90	.9962	.4981	.9981	.0037	.0060	.0150
2.95	.9968	.4984	.9984	.0032	.0051	.0129
3.00	.9973	.4987	.9987	.0027	.0044	.0111

Independent Variable = Probability = A_{xx}

x/σ	A_{xx}	A_{ox}	A_x	A_{tt}	y	y/y_o
.005	.005	.002	.502	.995	.399	.999
.013	.010	.005	.505	.990	.399	.999
.063	.050	.025	.525	.950	.398	.998
.126	.100	.050	.550	.900	.396	.990
.189	.150	.075	.575	.850	.392	.982
.253	.200	.100	.600	.800	.386	.965
.319	.250	.125	.625	.750	.379	.950

Table 1. Ordinates and Areas for Normal or Gaussian Probability Distribution (continued)

x/σ	A_{xx}	A_{ox}	A_x	A_{tt}	y	y/y_o
.385	.300	.150	.650	.700	.370	.925
.454	.350	.175	.675	.650	.360	.900
.524	.400	.200	.700	.600	.348	.870
.598	.450	.225	.725	.550	.334	.835
.674	.500	.250	.750	.500	.318	.795
.755	.550	.275	.775	.450	.300	.749
.842	.600	.300	.800	.400	.280	.702
.935	.650	.325	.825	.350	.258	.643
1.036	.700	.350	.850	.300	.233	.583
1.150	.750	.375	.875	.250	.206	.516
1.282	.800	.400	.900	.200	.176	.440
1.440	.850	.425	.925	.150	.142	.355
1.645	.900	.450	.950	.100	.103	.257
1.960	.950	.475	.975	.050	.058	.146
2.054	.960	.480	.980	.040	.048	.121
2.170	.970	.485	.985	.030	.038	.095
2.326	.980	.490	.990	.020	.027	.066
2.576	.990	.495	.995	.010	.014	.036
2.748	.995	.497	.997	.005	.009	.022
3.090	.999	.499	.999	.001	.003	.008

Table 2. Student's *t*-Distribution*

Values of *t* at Specified Levels of Significance; Residual Area A_{tt}, Two Tails

SYMBOLS: *N* = degrees of freedom *P* = probability of agreement

N \ *P*	.50	.40	.30	.20	.10	.05	.02	.01	.005	.001
1	1.000	1.376	1.963	3.078	6.314	12.706	31.821	63.657	127.32	636.619
2	.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	14.089	31.598
3	.765	.978	1.250	1.638	2.353	3.182	4.541	5.841	7.453	12.924
4	.741	.941	1.190	1.533	2.132	2.776	3.747	4.604	5.598	8.610
5	.727	.920	1.156	1.476	2.015	2.571	3.365	4.032	4.773	6.869
6	.718	.906	1.134	1.440	1.943	2.447	3.143	3.707	4.317	5.959
7	.711	.896	1.119	1.415	1.895	2.365	2.998	3.499	4.029	5.408
8	.706	.889	1.108	1.397	1.860	2.306	2.896	3.355	3.832	5.041
9	.703	.883	1.100	1.383	1.833	2.262	2.821	3.250	3.690	4.781
10	.700	.879	1.093	1.372	1.812	2.228	2.764	3.169	3.581	4.587
11	.697	.876	1.088	1.363	1.796	2.201	2.718	3.106	3.497	4.437
12	.695	.873	1.083	1.356	1.782	2.179	2.681	3.055	3.428	4.318
13	.694	.870	1.079	1.350	1.771	2.160	2.650	3.012	3.372	4.221
14	.692	.868	1.076	1.345	1.761	2.145	2.624	2.977	3.326	4.140
15	.691	.866	1.074	1.341	1.753	2.131	2.602	2.947	3.286	4.073
16	.690	.865	1.071	1.337	1.746	2.120	2.583	2.921	3.252	4.015
17	.689	.863	1.069	1.333	1.740	2.110	2.567	2.898	3.222	3.965
18	.688	.862	1.067	1.330	1.734	2.101	2.552	2.878	3.197	3.922
19	.688	.861	1.066	1.328	1.729	2.093	2.539	2.861	3.174	3.883
20	.687	.860	1.064	1.325	1.725	2.086	2.528	2.845	3.153	3.850
21	.686	.859	1.063	1.323	1.721	1.080	2.518	2.831	3.135	3.819
22	.686	.858	1.061	1.321	1.717	2.074	2.508	2.819	3.119	3.792
23	.685	.858	1.060	1.319	1.714	2.069	2.500	2.807	3.104	3.767
24	.685	.857	1.059	1.318	1.711	2.064	2.492	2.797	3.090	3.745
25	.684	.856	1.058	1.316	1.708	2.060	2.485	2.787	3.078	3.725
26	.684	.856	1.058	1.315	1.706	2.056	2.479	2.779	3.067	3.707
27	.684	.855	1.057	1.314	1.703	2.052	2.473	2.771	3.056	3.690
28	.683	.855	1.056	1.313	1.701	2.048	2.467	2.763	3.047	3.674
29	.683	.854	1.055	1.311	1.699	2.045	2.462	2.756	3.038	3.659
30	.683	.854	1.055	1.310	1.697	2.042	2.457	2.750	3.030	3.646
40	.681	.851	1.050	1.303	1.684	2.021	2.423	2.704	2.971	3.551
60	.679	.848	1.046	1.296	1.671	2.000	2.390	2.660	2.915	3.460
120	.677	.845	1.041	1.289	1.658	1.980	2.358	2.617	2.860	3.373
∞	.674	.842	1.036	1.282	1.645	1.960	2.326	2.576	2.807	3.291

* Abridged from *Statistical Tables for Biological, Agricultural, and Medical Research,* 6th ed., R.A. Fisher and F. Yates, published by Oliver and Boyd, by permission of the authors and publishers; and *Biometrika Tables for Statisticians,* E.S. Pearson and H.O. Hartley, Eds., Vol. 1, Cambridge University Press, 1962.

Table 3. Chi-Square Distribution*

Values of Chi Square at Specified Levels of Significance; Single Tail

SYMBOLS: *N* = degrees of freedom *P* = probability of agreement

N \ *P*	.995	.990	.975	.950	.900	.750	.500	.250	.100	.050	.025	.010	.005	.001	*N*
1	—	.0002	.001	.0039	.0158	.102	.455	1.32	2.71	3.84	5.02	6.63	7.88	10.83	1
2	.0100	.0201	.0506	.103	.211	.575	1.39	2.77	4.61	5.99	7.38	9.21	10.6	13.82	2
3	.0717	.115	.216	.352	.584	1.21	2.37	4.11	6.25	7.81	9.35	11.3	12.8	16.27	3
4	.207	.297	.484	.711	1.06	1.92	3.36	5.39	7.78	9.49	11.1	13.3	14.9	18.47	4
5	.412	.554	.831	1.15	1.61	2.67	4.35	6.63	9.24	11.1	12.8	15.1	16.7	20.52	5
6	.676	.872	1.24	1.64	2.20	3.45	5.35	7.84	10.6	12.6	14.4	16.8	18.5	22.46	6
7	.989	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.0	14.1	16.0	18.5	20.3	24.32	7
8	1.34	1.65	2.18	2.73	3.49	5.07	7.34	10.2	13.4	15.5	17.5	20.1	22.0	26.13	8
9	1.73	2.09	2.70	3.33	4.17	5.90	8.34	11.4	14.7	16.9	19.0	21.7	23.6	27.88	9
10	2.16	2.56	3.25	3.94	4.87	6.74	9.34	12.5	16.0	18.3	20.5	23.2	25.2	29.59	10
11	2.60	3.05	3.82	4.57	5.58	7.58	10.3	13.7	17.3	19.7	21.9	24.7	26.8	31.26	11
12	3.07	3.57	4.40	5.23	6.30	8.44	11.3	14.8	18.5	21.0	21.3	26.2	28.3	32.91	12
13	3.57	4.11	5.01	5.89	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8	34.53	13
14	4.07	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3	36.12	14
15	4.60	5.23	6.26	7.26	8.55	11.0	14.3	18.2	22.3	25.0	27.5	30.6	32.8	37.70	15
16	5.14	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3	39.25	16
17	5.70	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	30.2	33.4	35.7	40.79	17
18	6.26	7.01	8.23	9.39	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2	42.31	18
19	6.84	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6	43.82	19
20	7.43	8.26	9.59	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0	45.32	20
21	8.03	8.90	10.3	11.6	13.2	16.3	20.3	24.9	29.6	32.7	35.5	38.9	41.4	46.80	21
22	8.64	9.54	11.0	12.3	14.0	17.2	21.3	26.0	30.8	33.9	36.8	40.3	42.8	48.27	22
23	9.26	10.2	11.7	13.1	14.8	18.1	22.3	27.1	32.0	35.2	38.1	41.6	44.2	49.73	23
24	9.89	10.9	12.4	13.8	15.7	19.0	23.3	28.2	33.2	36.4	39.4	43.0	45.6	51.18	24
25	10.5	11.5	13.1	14.6	16.5	19.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9	52.62	25
26	11.2	12.2	13.8	15.4	17.3	20.8	25.3	30.4	35.6	38.9	41.9	45.6	48.3	54.05	26
27	11.8	12.9	14.6	16.2	18.1	21.7	26.3	31.5	36.7	40.1	43.2	47.0	49.6	55.48	27
28	12.5	13.6	15.3	16.9	18.9	22.7	27.3	32.6	37.9	41.3	44.5	48.3	51.0	56.89	28
29	13.1	14.3	16.0	17.7	19.8	23.6	28.3	33.7	39.1	42.6	45.7	49.6	52.3	58.30	29
30	13.8	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7	59.70	30

*From: *Biometrika Tables for Statisticians*, E.S. Pearson and H.O. Hartley, Eds., Vol. 1, Cambridge University Press, 1962.

Table 4. F-Distribution*

For m and n Degrees of Freedom; p = .001 to .100; Single Tail

Table A. p = .001

n \ m	1	2	3	4	5	6	7	8	9	10	15	30	60	∞
1	4053†	5000†	5404†	5625†	5764†	5859†	5929†	5981†	6023†	6056†	6158†	6261†	6313†	6366†
2	998.5	999.0	999.2	999.2	999.3	999.3	999.4	999.4	999.4	999.4	999.4	999.5	999.5	999.5
3	167.0	148.5	141.1	137.1	134.6	132.8	131.6	130.6	129.9	129.2	127.4	125.4	124.5	123.5
4	74.14	61.25	56.18	53.44	51.71	50.53	49.66	49.00	48.47	48.05	46.76	45.43	44.75	44.05
5	47.18	37.12	33.20	31.09	29.75	28.84	28.16	27.64	27.24	26.92	25.91	24.87	24.33	23.79
6	35.51	27.00	23.70	21.92	20.81	20.03	19.46	19.03	18.69	18.41	17.56	16.67	16.21	15.75
7	29.25	21.69	18.77	17.19	16.21	15.52	15.02	14.63	14.33	14.08	13.32	12.53	12.12	11.70
8	25.42	18.49	15.83	14.39	13.49	12.86	12.40	12.04	11.77	11.54	10.84	10.11	9.73	9.33
9	22.86	16.39	13.90	12.56	11.71	11.13	10.70	10.37	10.11	9.89	9.24	8.55	8.19	7.81
10	21.04	14.91	12.55	11.28	10.48	9.92	9.52	9.20	8.96	8.75	8.13	7.47	7.12	6.76
12	18.64	12.97	10.80	9.63	8.89	8.38	8.00	7.71	7.48	7.29	6.71	6.09	5.76	5.42
15	16.59	11.34	9.34	8.25	7.57	7.09	6.74	6.47	6.26	6.08	5.54	4.95	4.64	4.31
30	13.29	8.77	7.05	6.12	5.53	5.12	4.82	4.58	4.39	4.24	3.75	3.22	2.92	2.59
60	11.97	7.76	6.17	5.31	4.76	4.37	4.09	3.87	3.69	3.54	3.08	2.55	2.25	1.89
∞	10.83	6.91	5.42	4.62	4.10	3.74	3.47	3.27	3.10	2.96	2.51	1.99	1.66	1.00

†Multiply these entries by 100.

Table B. p = .005

n \ m	1	2	3	4	5	6	7	8	9	10	15	30	60	∞
1	16211	20000	21615	22500	23056	24437	23715	23925	24091	24630	24224	25044	25253	25465
2	198.5	199.0	199.2	199.2	199.3	199.3	199.4	199.4	199.4	199.4	199.4	199.5	199.5	199.5
3	55.55	49.80	47.47	46.19	45.39	44.84	44.43	44.13	43.88	43.69	43.08	42.47	42.15	41.83
4	31.33	26.28	24.26	23.15	22.46	21.97	21.62	21.35	21.14	20.97	20.44	19.89	19.61	19.32
5	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.15	12.66	12.40	12.14
6	18.63	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	9.81	9.36	9.12	8.88
7	16.24	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.51	8.38	7.97	7.53	7.31	7.08
8	14.69	11.04	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	6.81	6.40	6.18	5.95
9	13.61	10.11	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.03	5.62	5.41	5.19
10	12.83	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.47	5.07	4.86	4.64

Table 4. F-Distribution* (continued)

n \ m	1	2	3	4	5	6	7	8	9	10	15	30	60	∞
12	11.75	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.72	4.33	4.12	3.90
15	10.80	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.07	3.69	3.48	3.26
30	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.01	2.63	2.42	2.18
60	8.49	5.79	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.57	2.19	1.96	1.69
∞	7.88	5.30	4.28	3.72	3.35	3.09	2.90	2.74	2.62	2.52	2.19	1.79	1.53	1.00

Table C. $p = .010$

n \ m	1	2	3	4	5	6	7	8	9	10	15	30	60	∞
1	4052	4999.5	5403	5625	5764	5859	5928	5982	6022	6056	6157	6261	6313	6366
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.43	99.47	99.48	99.50
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	26.87	26.50	26.32	26.13
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.20	13.84	13.65	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.72	9.38	9.20	9.02
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.56	7.23	7.06	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.31	5.99	5.82	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.52	5.20	5.03	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.96	4.65	4.48	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.56	4.25	4.08	3.91
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.01	3.70	3.54	3.36
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.52	3.21	3.05	2.87
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.70	2.39	2.21	2.01
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.35	2.03	1.84	1.60
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.04	1.70	1.47	1.00

Table D. $p = .025$

n \ m	1	2	3	4	5	6	7	8	9	10	15	30	60	∞
1	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	963.3	968.6	984.9	1001	1010	1018
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.43	39.46	39.48	39.50

3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.25	14.08	13.99	13.90
4	12.22	10.65	9.98	6.60	9.36	9.20	9.07	8.98	8.90	8.84	8.66	8.46	8.36	8.26
5	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.43	6.23	6.12	6.02
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.27	5.07	4.96	4.85
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.57	4.36	4.25	4.14
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.10	3.89	3.78	3.67
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.77	3.56	3.45	3.33
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.52	3.31	3.20	3.08
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.18	2.96	2.85	2.72
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.86	2.64	2.52	2.40
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.31	2.07	1.94	1.79
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.06	1.82	1.67	1.48
∞	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.83	1.57	1.39	1.00

Table E. p = .050

n \ m	1	2	3	4	5	6	7	8	9	10	15	30	60	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	245.9	250.1	252.2	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.43	19.46	19.48	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.70	8.62	8.57	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.86	5.75	5.69	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.62	4.50	4.43	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	3.94	3.81	3.74	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.51	3.38	3.30	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.22	3.08	3.01	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.01	2.86	2.79	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.85	2.70	2.62	2.54
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.62	2.47	2.38	2.30
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.40	2.25	2.16	2.07
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.01	1.84	1.74	1.62
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.84	1.65	1.53	1.39
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.67	1.46	1.32	1.00

Table 4. F-Distribution* (continued)

Table F. p = .100

n \ m	1	2	3	4	5	6	7	8	9	10	15	30	60	∞
1	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	59.86	60.19	61.22	62.26	62.79	63.33
2	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.42	9.46	9.47	9.49
3	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.20	5.17	5.15	5.13
4	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.87	3.82	3.79	3.76
5	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.24	3.17	3.14	3.10
6	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.87	2.80	2.76	2.72
7	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.63	2.56	2.51	2.47
8	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.46	2.38	2.34	2.29
9	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.34	2.25	2.21	2.16
10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.24	2.16	2.11	2.06
12	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.10	2.01	1.96	1.90
15	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	1.97	1.87	1.82	1.76
30	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.72	1.61	1.54	1.46
60	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.60	1.48	1.40	1.29
∞	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.49	1.34	1.24	1.00

*From *Biometrika Tables for Statisticians,* E.S. Pearson and H.O. Hartley, Eds., Vol. 1, Cambridge University Press, 1962.

Table 5. Binomial Distribution—Cumulative Probabilities: *P**

SYMBOLS:
N = number of trials or size of a sample
p = probability of the outcome for the entire population (success or failure, whichever is less)
P = cumulative probability of observing *X* or more successes within *N*

N	X	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
2	1	.0975	.1900	.2775	.3600	.4375	.5100	.5775	.6400	.6975	.7500
	2	.0025	.0100	.0225	.0400	.0625	.0900	.1225	.1600	.2025	.2500
3	1	.1426	.2710	.3859	.4880	.5781	.6570	.7254	.7840	.8336	.8750
	2	.0072	.0280	.0608	.1040	.1562	.2160	.2818	.3520	.4252	.5000
	3	.0001	.0010	.0034	.0080	.0156	.0270	.0429	.0640	.0911	.1250
4	1	.1855	.3439	.4780	.5904	.6836	.7599	.8215	.8704	.9085	.9375
	2	.0140	.0523	.1095	.1808	.2617	.3483	.4370	.5248	.6090	.6875
	3	.0005	.0037	.0120	.0272	.0508	.0837	.1265	.1792	.2415	.3125
	4	.0000	.0001	.0005	.0016	.0039	.0081	.0150	.0256	.0410	.0625
5	1	.2262	.4095	.5563	.6723	.7627	.8319	.8840	.9222	.9497	.9688
	2	.0226	.0815	.1648	.2627	.3672	.4718	.5716	.6630	.7438	.8125
	3	.0012	.0086	.0266	.0579	.1035	.1631	.2352	.3174	.4069	.5000
	4	.0000	.0005	.0022	.0067	.0156	.0308	.0540	.0870	.1312	.1875
	5	.0000	.0000	.0001	.0003	.0010	.0024	.0053	.0102	.0185	.0312
6	1	.2649	.4686	.6229	.7379	.8220	.8824	.9246	.9533	.9723	.9844
	2	.0328	.1143	.2235	.3447	.4661	.5798	.6809	.7667	.8364	.8906
	3	.0022	.0158	.0473	.0989	.1694	.2557	.3529	.4557	.5585	.6562
	4	.0001	.0013	.0059	.0170	.0376	.0705	.1174	.1792	.2553	.3438
	5	.0000	.0001	.0004	.0016	.0046	.0109	.0223	.0410	.0692	.1094
	6	.0000	.0000	.0000	.0001	.0002	.0007	.0018	.0041	.0083	.0156
7	1	.3017	.5217	.6794	.7903	.8665	.9176	.9510	.9720	.9848	.9922
	2	.0444	.1497	.2834	.4233	.5551	.6706	.7662	.8414	.8976	.9375
	3	.0038	.0257	.0738	.1480	.2436	.3529	.4677	.5801	.6836	.7734
	4	.0002	.0027	.0121	.0333	.0706	.1260	.1998	.2898	.3917	.5000
	5	.0000	.0002	.0012	.0047	.0129	.0288	.0556	.0963	.1529	.2266
	6	.0000	.0000	.0001	.0004	.0013	.0038	.0090	.0188	.0357	.0625
8	1	.3366	.5695	.7275	.8322	.8999	.9424	.9681	.9832	.9916	.9961
	2	.0572	.1869	.3428	.4967	.6329	.7447	.8309	.8936	.9368	.9648
	3	.0058	.0381	.1052	.2031	.3215	.4482	.5722	.6846	.7799	.8555
	4	.0004	.0050	.0214	.0563	.1138	.1941	.2936	.4059	.5230	.6367
	5	.0000	.0004	.0029	.0104	.0273	.0580	.1061	.1737	.2604	.3633
	6	.0000	.0000	.0002	.0012	.0042	.0113	.0253	.0498	.0885	.1445
	7	.0000	.0000	.0000	.0001	.0004	.0013	.0036	.0085	.0181	.0352
9	1	.3698	.6126	.7684	.8658	.9249	.9596	.9793	.9899	.9954	.9980
	2	.0712	.2252	.4005	.5638	.6997	.8040	.8789	.9295	.9615	.9805
	3	.0084	.0530	.1409	.2618	.3993	.5372	.6627	.7682	.8505	.9102
	4	.0006	.0083	.0339	.0856	.1657	.2703	.3911	.5174	.6386	.7461
	5	.0000	.0009	.0056	.0196	.0489	.0988	.1717	.2666	.3786	.5000
	6	.0000	.0001	.0006	.0031	.0100	.0253	.0536	.0994	.1658	.2539
	7	.0000	.0000	.0000	.0003	.0013	.0043	.0112	.0250	.0498	.0898
	8	.0000	.0000	.0000	.0000	.0001	.0004	.0014	.0038	.0091	.0195
10	1	.4013	.6513	.8031	.8926	.9437	.9718	.9865	.9940	.9975	.9990
	2	.0861	.2639	.4557	.6242	.7560	.8507	.9140	.9536	.9767	.9893
	3	.0115	.0702	.1798	.3222	.4744	.6172	.7384	.8327	.9004	.9453

Table 5. Binomial Distribution—Cumulative Probabilities: *P** (continued)

N	*X*	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
	4	.0010	.0128	.0500	.1209	.2241	.3504	.4862	.6177	.7340	.8281
	5	.0001	.0016	.0099	.0328	.0781	.1503	.2485	.3669	.4956	.6230
	6	.0000	.0001	.0014	.0064	.0197	.0473	.0949	.1662	.2616	.3770
	7	.0000	.0000	.0001	.0009	.0035	.0106	.0260	.0548	.1020	.1719
	8	.0000	.0000	.0000	.0001	.0094	.0016	.0048	.0123	.0274	.0547
	9	.0000	.0000	.0000	.0000	.0000	.0001	.0005	.0017	.0045	.0107
12	1	.4596	.7176	.8578	.9313	.9683	.9862	.9943	.9978	.9992	.9998
	2	.1184	.3410	.5565	.7251	.8416	.9150	.9576	.9804	.9917	.9968
	3	.0196	.1109	.2642	.4417	.6093	.7472	.8487	.9166	.9579	.9807
	4	.0022	.0256	.0922	.2054	.3512	.5075	.6533	.7747	.8655	.9270
	5	.0002	.0043	.0239	.0726	.1576	.2763	.4167	.5618	.6956	.8062
	6	.0000	.0005	.0046	.0194	.0544	.1178	.2127	.3348	.4731	.6128
	7	.0000	.0001	.0007	.0039	.0143	.0386	.0846	.1582	.2607	.3872
	8	.0000	.0000	.0001	.0006	.0028	.0095	.0255	.0573	.1117	.1938
	9	.0000	.0000	.0000	.0001	.0004	.0017	.0056	.0153	.0356	.0730
	10	.0000	.0000	.0000	.0000	.0000	.0002	.0008	.0028	.0079	.0193
15	1	.5367	.7941	.9126	.9648	.9866	.9953	.9984	.9995	.9999	1.0000
	2	.1710	.4510	.6814	.8329	.9198	.9647	.9858	.9948	.9983	.9995
	3	.0362	.1841	.3958	.6020	.7639	.8732	.9383	.9729	.9893	.9963
	4	.0055	.0556	.1773	.3518	.5387	.7031	.8273	.9095	.9576	.9824
	5	.0006	.0127	.0617	.1642	.3135	.4845	.6481	.7827	.8796	.9408
	6	.0001	.0022	.0168	.0611	.1484	.2784	.4357	.5968	.7392	.8491
	7	.0000	.0003	.0036	.0181	.0566	.1311	.2452	.3902	.5478	.6964
	8	.0000	.0000	.0006	.0042	.0173	.0500	.1132	.2131	.3465	.5000
	9	.0000	.0000	.0001	.0008	.0042	.0152	.0422	.0950	.1818	.3036
	10	.0000	.0000	.0000	.0001	.0008	.0037	.0124	.0338	.0769	.1509
	11	.0000	.0000	.0000	.0000	.0001	.0007	.0028	.0093	.0255	.0592
	12	.0000	.0000	.0000	.0000	.0000	.0001	.0005	.0019	.0063	.0176
20	1	.6415	.8784	.9612	.9885	.9968	.9992	.9998	1.0000	1.0000	1.0000
	2	.2642	.6083	.8244	.9308	.9757	.9924	.9979	.9995	.9999	1.0000
	3	.0755	.3231	.5951	.7939	.9087	.9645	.9879	.9964	.9991	.9998
	4	.0159	.1330	.3523	.5886	.7748	.8929	.9556	.9840	.9951	.9987
	5	.0026	.0432	.1702	.3704	.5852	.7625	.8818	.9490	.9811	.9941
	6	.0003	.0113	.0673	.1958	.3828	.5836	.7546	.8744	.9447	.9793
	7	.0000	.0024	.0219	.0867	.2142	.3920	.5834	.7500	.8701	.9423
	8	.0000	.0004	.0059	.0321	.1018	.2277	.3990	.5841	.7480	.8684
	9	.0000	.0001	.0013	.0100	.0409	.1133	.2376	.4044	.5857	.7483
	10	.0000	.0000	.0002	.0026	.0139	.0480	.1218	.2447	.4086	.5881
	11	.0000	.0000	.0000	.0006	.0039	.0171	.0532	.1275	.2493	.4119
	12	.0000	.0000	.0000	.0001	.0009	.0051	.0196	.0565	.1308	.2517
	13	.0000	.0000	.0000	.0000	.0002	.0013	.0060	.0210	.0580	.1316
	14	.0000	.0000	.0000	.0000	.0000	.0003	.0015	.0065	.0214	.0577
	15	.0000	.0000	.0000	.0000	.0000	.0000	.0003	.0016	.0064	.0207

Note: Individual binomial probability terms can be obtained by subtraction, i.e.,

$$p_x = \sum_x^N (p_x) - \sum_{x+1}^N (p_x)$$

* Condensed from: *CRC Handbook of Tables for Mathematics,* 4th ed., S.M. Selby, Ed., The Chemical Rubber Co., 1970.

Table 6. Poisson Distribution—Cumulative Probabilities: P*

SYMBOLS:
N = number of trials or size of the sample
p = fixed probability of the outcome for entire population (success or failure, whichever is less)
P = cumulaive probability of X or more successes (or failures) within N

X′ \ Np	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
1	.095	.181	.259	.330	.394	.451	.503	.551	.593	.632	.667	.699
2	.005	.018	.037	.062	.090	.122	.156	.191	.228	.264	.301	.337
3	.000	.001	.004	.008	.014	.023	.034	.047	.063	.080	.100	.121
4	.000	.000	.000	.001	.002	.003	.006	.0009	.014	.019	.026	.034

X′ \ Np	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4
1	.728	.753	.777	.798	.817	.835	.850	.865	.878	.889	.900	.909
2	.373	.408	.442	.475	.507	.537	.566	.594	.620	.645	.669	.692
3	.143	.167	.191	.217	.243	.269	.296	.323	.350	.377	.404	.430
4	.043	.054	.066	.079	.093	.109	.125	.143	.161	.181	.201	.221
5	.011	.014	.019	.024	.030	.036	.044	.053	.062	.073	.084	.096
6	.002	.003	.005	.006	.008	.010	.013	.017	.020	.025	.030	.036

X′ \ Np	2.5	2.6	2.7	2.8	2.9	3.0	3.1	3.2	3.3	3.4	3.5	3.6
1	.918	.926	.933	.939	.945	.950	.955	.959	.963	.967	.970	.973
2	.713	.733	.751	.769	.785	.801	.815	.829	.841	.853	.864	.874
3	.456	.482	.506	.531	.554	.577	.599	.620	.641	.660	.679	.697
4	.242	.264	.286	.308	.330	.353	.375	.398	.420	.442	.463	.485
5	.109	.123	.137	.152	.168	.185	.202	.219	.237	.256	.275	.294
6	.042	.049	.057	.065	.074	.084	.094	.105	.117	.130	.142	.156
7	.014	.017	.021	.024	.029	.034	.039	.045	.051	.058	.065	.073
8	.004	.005	.007	.008	.010	.012	.014	.017	.020	.023	.027	.031

X′ \ Np	3.7	3.8	3.9	4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8
1	.975	.978	.980	.982	.983	.985	.986	.988	.989	.990	.991	.992
2	.884	.893	.901	.908	.916	.922	.928	.934	.939	.944	.948	.952
3	.715	.731	.747	.762	.776	.790	.803	.815	.826	.837	.848	.858
4	.506	.527	.547	.567	.586	.605	.623	.641	.658	.674	.690	.706
5	.313	.332	.352	.371	.391	.410	.430	.449	.468	.487	.505	.524
6	.170	.184	.199	.215	.231	.247	.263	.280	.297	.314	.332	.349
7	.082	.091	.101	.111	.121	.133	.144	.156	.169	.182	.195	.209
8	.035	.040	.045	.051	.057	.064	.071	.079	.087	.095	.104	.113
9	.014	.016	.019	.021	.025	.028	.032	.036	.040	.045	.050	.056
10	.005	.006	.007	.008	.010	.011	.013	.015	.017	.020	.022	.025

X′ \ Np	4.9	5.0	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9	6.0
1	.993	.993	.994	.995	.995	.996	.996	.996	.997	.997	.997	.998
2	.956	.960	.963	.966	.969	.971	.973	.976	.978	.979	.981	.983
3	.867	.875	.884	.891	.898	.905	.912	.918	.923	.929	.933	.938
4	.721	.735	.749	.762	.775	.787	.798	.809	.820	.830	.840	.849
5	.542	.560	.577	.594	.611	.627	.643	.658	.673	.687	.701	.715
6	.367	.384	.402	.419	.437	.454	.471	.488	.505	.522	.538	.554
7	.223	.238	.253	.268	.283	.298	.314	.330	.346	.362	.378	.394
8	.123	.133	.144	.155	.167	.178	.191	.203	.216	.229	.242	.256

Table 6. Poisson Distribution—Cumulative Probabilities: P^* (continued)

X′ \ Np	4.9	5.0	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9	6.0
9	.062	.068	.075	.082	.089	.097	.106	.114	.123	.133	.143	.153
10	.028	.032	.036	.040	.044	.049	.054	.059	.065	.071	.077	.084
11	.012	.014	.016	.018	.020	.023	.025	.028	.031	.035	.039	.043
12	.005	.006	.006	.007	.008	.010	.011	.013	.014	.016	.018	.020

X′ \ Np	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9	7.0	7.1	7.2
1	.998	.998	.998	.998	.999	.999	.999	.999	.999	.999	.999	.999
2	.984	.985	.987	.988	.989	.990	.991	.991	.992	.993	.993	.994
3	.942	.946	.950	.954	.957	.960	.963	.966	.968	.970	.973	.975
4	.858	.866	.874	.881	.888	.895	.901	.907	.913	.918	.923	.928
5	.728	.741	.753	.765	.776	.787	.798	.808	.818	.827	.836	.845
6	.570	.586	.601	.616	.631	.645	.659	.673	.686	.699	.712	.724
7	.410	.426	.442	.458	.474	.489	.505	.520	.535	.550	.565	.580
8	.270	.284	.298	.313	.327	.342	.357	.372	.386	.401	.416	.431
9	.163	.174	.185	.197	.208	.220	.233	.245	.258	.271	.284	.297
10	.091	.098	.106	.114	.123	.131	.140	.150	.151	.170	.180	.190
11	.047	.051	.056	.061	.067	.073	.079	.085	.092	.099	.106	.113
12	.022	.025	.028	.031	.034	.037	.041	.045	.049	.053	.058	.063

X′ \ Np	7.3	7.4	7.5	7.6	7.7	7.8	7.9	8.0	8.2	8.4	8.6	8.8
1	.999	.999	.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	.994	.995	.995	.996	.996	.996	.997	.997	.998	.998	.998	.999
3	.976	.978	.980	.981	.983	.984	.985	.986	.988	.990	.991	.993
4	.933	.937	.941	.945	.948	.952	.955	.958	.963	.968	.972	.976
5	.853	.861	.868	.875	.882	.888	.895	.900	.911	.921	.930	.938
6	.736	.747	.759	.769	.780	.790	.799	.809	.826	.843	.858	.872
7	.594	.608	.622	.635	.649	.662	.674	.687	.710	.733	.754	.774
8	.446	.461	.475	.490	.504	.519	.533	.547	.575	.601	.627	.652
9	.311	.324	.338	.352	.366	.380	.394	.408	.435	.463	.491	.518
10	.201	.212	.224	.235	.247	.259	.271	.283	.309	.334	.360	.386
11	.121	.129	.138	.147	.156	.165	.174	.184	.205	.226	.248	.271
12	.068	.074	.079	.085	.092	.098	.105	.112	.127	.143	.160	.178
13	.036	.039	.043	.046	.050	.055	.059	.064	.074	.085	.097	.110
14	.018	.020	.022	.024	.026	.029	.031	.034	.041	.048	.056	.064

X′ \ Np	9.0	10	11	12	13	14	15	16	17	18	19	20
1	.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.1.000
3	.994	.997	.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.1.000
4	.979	.990	.995	.998	.999	1.000	1.000	1.000	1.000	1.000	1.000	.1.000
5	.945	.971	.985	.992	.996	.998	.999	1.000	1.000	1.000	1.000	.1.000
6	.884	.933	.963	.980	.989	.995	.997	.999	.999	1.000	1.000	.1.000
7	.793	.870	.921	.954	.974	.986	.992	.996	.998	.999	1.000	1.000
8	.676	.780	.857	.911	.946	.968	.982	.990	.995	.997	.999	.999
9	.544	.667	.768	.845	.900	.938	.963	.978	.987	.993	.996	.998
10	.413	.542	.660	.758	.834	.891	.930	.957	.974	.985	.991	.995
11	.294	.417	.540	.653	.748	.824	.882	.923	.951	.970	.982	.989
12	.197	.303	.421	.538	.647	.740	.815	.873	.915	.945	.965	.979

Table 6. Poisson Distribution—Cumulative Probabilities: P* (continued)

X' \ Np	9.0	10	11	12	13	14	15	16	17	18	19	20
13	.124	.208	.311	.424	.537	.642	.732	.807	.865	.908	.939	.961
14	.074	.136	.219	.319	.427	.536	.637	.726	.799	.857	.902	.934
15	.042	.084	.146	.228	.325	.430	.534	.633	.719	.792	.850	.895
16	.022	.049	.093	.156	.236	.331	.432	.533	.629	.713	.785	.844
17	.011	.027	.056	.101	.165	.244	.336	.434	.532	.625	.708	.779
18	.005	.014	.032	.063	.110	.173	.251	.341	.436	.531	.622	.703
19	.002	.007	.018	.037	.070	.117	.181	.258	.345	.438	.531	.619
20	.001	.004	.009	.021	.043	.077	.125	.188	.264	.349	.439	.530
21	.000	.002	.005	.012	.025	.048	.083	.132	.195	.269	.353	.441
22	.000	.001	.002	.006	.014	.029	.053	.089	.139	.201	.275	.356
23	.000	.000	.001	.003	.008	.017	.033	.058	.095	.145	.207	.279
24	.000	.000	.001	.002	.004	.009	.020	.037	.063	.101	.151	.213
25	.000	.000	.000	.001	.002	.005	.011	.022	.041	.068	.107	.157
26	.000	.000	.000	.000	.001	.003	.006	.013	.025	.045	.073	.112

Note: Individual Poisson-probability terms can be obtained by subtraction, i.e.,

$$p_x = \sum_x^N (p_x) - \sum_{x+1}^N (p_x)$$

* Condensed from *CRC Handbook of Tables for Mathematics,* 4th ed., S.M. Selby, Ed., The Chemical Rubber Co., 1970.

Critical Values for the Sign Test

The observations in a random sample of size n from X and those of the same size from Y are paired according to the order of observation: (X_i, Y_i), $i = 1, 2,\ldots, n$. The differences $d_i = X_i - Y_i$ are calculated for each of the n pairs. the null hypothesis is that the difference d_i has a distribution with median zero, i.e., the true proportion of positive (negative) signs is equal to $p = \frac{1}{2}$. Thus the test is whether X and Y have the same median. The probability of x positive (negative) signs is given by the binomial probability function

$$f(x) = f\left(x; n, p = \frac{1}{2}\right) = \binom{n}{x}\left(\frac{1}{2}\right)^n$$

This table gives the critical value k such that

$$P(x \leq k) = \sum_{x=0}^{k} \binom{n}{x}\left(\frac{1}{2}\right)^n < \frac{\alpha}{2}$$

Two-tail Percentage Points for the Binomial for $p = .5$

n	1%	5%	10%	25%	n	1%	5%	10%	25%
1					46	13	15	16	18
2					47	14	16	17	19
3				0	48	14	16	17	19
4				0	49	15	17	18	19
5			0	0	50	15	17	18	20
6		0	0	1	51	15	18	19	20
7		0	0	1	52	16	18	19	21
8	0	0	1	1	53	16	18	20	21
9	0	1	1	2	54	17	19	20	22
10	0	1	1	2	55	17	19	20	22
11	0	1	2	3	56	17	20	21	23
12	1	2	2	3	57	18	20	21	23
13	1	2	3	3	58	18	21	22	24
14	1	2	3	4	59	19	21	22	24
15	2	3	3	4	60	19	21	23	25
16	2	3	4	5	61	20	22	23	25
17	2	4	4	5	62	20	22	24	25
18	3	4	5	6	63	20	23	24	26
19	3	4	5	6	64	21	23	24	26
20	3	5	5	6	65	21	24	25	27
21	4	5	6	7	66	22	24	25	27
22	4	5	6	7	67	22	25	26	28
23	4	6	7	8	68	22	25	26	28
24	5	6	7	8	69	23	25	27	29
25	5	7	7	9	70	23	26	27	29
26	6	7	8	9	71	24	26	28	30
27	6	7	8	10	72	24	27	28	30
28	6	8	9	10	73	25	27	28	31
29	7	8	9	10	74	25	28	29	31
30	7	8	10	11	75	25	28	29	32
31	7	9	10	11	76	26	28	30	32
32	8	9	10	12	77	26	29	30	32
33	8	10	11	12	78	27	29	31	33
34	9	10	11	13	79	27	30	31	33
35	9	11	12	13	80	28	30	32	34
36	9	11	12	14	81	28	31	32	34
37	10	12	13	14	82	28	31	33	35
38	10	12	13	14	83	29	32	33	35
39	11	12	13	15	84	29	32	33	36
40	11	13	14	15	85	30	32	34	36
41	11	13	14	16	86	30	33	34	37
42	12	14	15	16	87	31	33	35	37
43	12	14	15	17	88	31	34	35	38
44	13	15	16	17	89	31	34	36	38
45	13	15	16	18	90	32	35	36	39

For values of n larger than 90, approximate values of r may be found by taking the nearest integer less than $(n - 1)/2 - k\sqrt{n + 1}$, where k is 1.2879, 0.9800, 0.8224, 0.5752 for the 1, 5, 10, 25% values, respectively.

From Bolz, R.E. and Tuve, G.L., Mathematical and statistical tables, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, p. 935.

Factors for Computing Control Limits

A. Control Charts for Measurement

If the process mean and standard deviation, μ and σ, are known, and it is assumed that the underlying distribution is normal, it is possible to assert with probability $1 - \alpha$ that the mean of a random sample of size n will fall between $\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ and $\bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$. These two limits on $\bar{x}$ provide upper and lower control limits. In actual practice μ and σ are usually unknown, and it is necessary to estimate their values from a large sample taken while the process is "in control." The central line of an $\bar{x}$ chart is given by μ, and the lower and upper three-sigma control limits are given by $\mu - A\sigma$ and $\mu + A\sigma$, respectively, where $A = \frac{3}{\sqrt{n}}$ and n is the sample size. Where the population parameters are unknown, it is necessary to estimate these parameters on the basis of preliminary samples. If k samples are used, each of size n, denote the mean of the ith sample by $\bar{x}_i$ and the grand mean of the k sample means by $\bar{\bar{x}}$, i.e.,

$$\bar{\bar{x}} = \frac{1}{k}\sum_{i=1}^{k} \bar{x}_i$$

Denote the range of the ith sample by R_i and by $\bar{R}$ the mean of the k sample ranges, i.e.,

$$\bar{R} = \frac{1}{k}\sum_{i=1}^{k} R_i$$

Since $\bar{x}$ is an unbiased estimate of the population mean μ, the central line for the $\bar{x}$ chart is given by $\bar{\bar{x}}$. The statistic R does not provide an unbiased estimate of σ, but $A_2\bar{R}$ is an unbiased estimate of $\frac{3\sigma}{\sqrt{n}}$. The constant multiplier A_2 depends on the assumption of normality. Thus, the central line and the lower and upper three-sigma limits, LCL and UCL, for an x chart (with μ and σ estimated from past data) are given by

$$\textit{central line} = \bar{\bar{x}}$$

$$LCL = \bar{\bar{x}} - A_2\bar{R}$$

$$UCL = \bar{\bar{x}} + A_2\bar{R}$$

The central line and control limits of an R chart are based on the distribution of the range of samples of size n from a normal population. The mean and standard deviation of the sampling distribution of R are given by $d_2\sigma$ and $d_3\sigma$, respectively, when σ is known. Here d_2 and d_3 are constants that depend on the size of the sample. The set of control-chart values for an R chart (with σ known) is given by

$$\textit{central line} = d_2\sigma$$

$$LCL = D_1\sigma$$

$$UCL = D_2\sigma$$

where $D_1 = d_2 - 3d_3$ and $D_2 = d_2 + 3d_3$.

If σ unknown, the control chart values for an R chart are given by

$$\textit{central line} = \bar{R}$$

$$LCL = D_3\bar{R}$$

$$UCL = D_4\bar{R}$$

where $D_3 = \frac{D_1}{d_2}$ and $D_4 = \frac{D_2}{d_2}$.

The central line and control limits of an s chart are based on estimates obtained from the samples. A pooled estimate of the population variance is obtained from the k samples, i.e., Factors for Computing Control Limits (continued)

$$s_p^2 = \frac{\sum_i (n_i - 1)s_i^2}{\sum_i (n_i - 1)}, \quad i = 1, 2, \ldots, k$$

If the sample sizes are all equal, the pooled estimate is

$$s_p^2 = \frac{1}{k}\sum_i s_i^2$$

The control chart values for an s chart are given by

$$\begin{aligned} \textit{central line } &= C_2' s_p \\ LCL &= B_2' s_p \\ UCL &= B_4' s_p \end{aligned}$$

If one uses the biased estimator of the variance s_p', as is often done in quality control work, the control chart values are given by

$$\begin{aligned} \textit{central line } &= c_2 s_p' \\ LCL &= B_2 s_p' \\ UCL &= B_4 s_p' \end{aligned}$$

B. Control Charts for Attributes

Control limits for a fraction-defective chart are based on the sampling theory for proportions, using the normal curve approximation to the binomial. If k samples are taken, the estimator of p is given by

$$\bar{p} = \frac{\sum_i x_i}{\sum_i n_i}, \quad i = 1, 2, \ldots, k$$

where x_i is the number of defectives in the ith sample of size n_i. The cenral line and control limits of a fraction-defective chart based on analysis of past data are given by

$$\begin{aligned} \textit{central line } &= \bar{p} \\ LCL &= \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n_i}} \\ UCL &= \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n_i}} \end{aligned}$$

When the sample sizes are approximately equal, n_i is replaced by $\bar{n} = \frac{1}{k}\sum_i n_i$.

Equivalent to the p chart for the fraction defective is the control chart for the number of defective. Here, if p is estimated by $\bar{p}$, the control-chart values for a number-of-defectives chart are given by

$$\begin{aligned} \textit{central line } &= \bar{n}\,\bar{p} \\ LCL &= \bar{n}\,\bar{p} - 3\sqrt{\bar{n}\,\bar{p}(1-\bar{p})} \\ UCL &= \bar{n}\,\bar{p} + 3\sqrt{\bar{n}\,\bar{p}(1-\bar{p})} \end{aligned}$$

Factors for Computing Control Limits (continued)

In many cases it is necessary to control the number of defects per unit C, where C is taken to be a value of a random variable having a Poisson distribution. If k is the number of units available for estimating λ, the parameter of the Poisson distribution, and if C_i is the number of defects in the ith unit, than λ is estimated by

$$\bar{C} = \frac{1}{k}\sum_{i=1}^{k} C_i$$

and the control-chart values for the C chart are

$$\text{central line } = \bar{C}$$

$$LCL = \bar{C} - 3\sqrt{\bar{C}}$$

$$UCL = \bar{C} + 3\sqrt{\bar{C}}$$

This table presents values of the factors for computing control limis for various sample sizes n.

Number of Observations in Sample, n	$\bar{X}$ Chart		R Chart			s Chart			σ Chart (biased)		
	Factors for Control Limits		Factor for Central Line	Factors for Control Limits		Factor for Central Line	Factors for Control Limits		Factor for Central Line	Factors for Control Limits	
	A	A_2	d_2	D_3	D_4	c'_2	B'_2	B'_4	c_2	B_2	B_4
2	2.121	1.880	1.128	0	3.267	0.798	0	2.298	0.5642	0	3.267
3	1.732	1.023	1.693	0	2.575	0.886	0	2.111	0.7236	0	2.568
4	1.500	0.729	2.059	0	2.282	0.921	0	1.982	0.7979	0	2.266
5	1.342	0.577	2.326	0	2.115	0.940	0	1.889	0.8407	0	2.089
6	1.225	0.483	2.534	0	2.004	0.951	0.085	1.817	0.8686	0.030	1.970
7	1.134	0.419	2.704	0.076	1.924	0.960	0.158	1.762	0.8882	0.118	1.882
8	1.061	0.373	2.847	0.136	1.864	0.965	0.215	1.715	0.9027	0.185	1.815
9	1.000	0.337	2.970	0.184	1.816	0.969	0.262	1.676	0.9139	0.239	1.761
10	0.949	0.308	3.078	0.223	1.777	0.973	0.302	1.644	0.9227	0.284	1.716
11	0.905	0.285	3.173	0.256	1.744	0.976	0.336	1.616	0.9300	0.321	1.679
12	0.866	0.266	3.258	0.284	1.716	0.977	0.365	1.589	0.9359	0.354	1.646
13	0.832	0.249	3.336	0.308	1.692	0.980	0.392	1.568	0.9410	0.382	1.618
14	0.802	0.235	3.407	0.329	1.671	0.981	0.414	1.548	0.9453	0.406	1.594
15	0.775	0.223	3.472	0.348	1.652	0.982	0.434	1.530	0.9490	0.428	1.572
16	0.750	0.212	3.532	0.364	1.636	0.984	0.454	1.514	0.9523	0.448	1.552
17	0.728	0.203	3.588	0.379	1.621	0.984	0.469	1.499	0.9551	0.466	1.534
18	0.707	0.194	3.640	0.392	1.608	0.986	0.486	1.486	0.9576	0.482	1.518
19	0.688	0.187	3.689	0.404	1.596	0.986	0.500	1.472	0.9599	0.497	1.503
20	0.671	0.180	3.735	0.414	1.586	0.987	0.513	1.461	0.9619	0.510	1.490
21	0.655	0.173	3.778	0.425	1.575	0.988	0.525	1.451	0.9638	0.523	1.477
22	0.640	0.167	3.819	0.434	1.566	0.988	0.536	1.440	0.9655	0.534	1.466
23	0.626	0.162	3.858	0.443	1.557	0.989	0.546	1.432	0.9670	0.545	1.455
24	0.612	0.157	3.895	0.452	1.548	0.989	0.556	1.422	0.9864	0.555	1.445
25	0.600	0.153	3.931	0.459	1.541	0.990	0.566	1.414	0.9696	0.565	1.435

From Bolz, R.E. and Tuve, G.L., Mathematical and statistical tables, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, p. 935.

Number Systems and Change of Base

Positional Notation

In our ordinary system of writing numbers, the value of any digit depends on its position in the number. The value of a digit in any position is ten times the value of the same digit one position to the right, or one-tenth the value of the same digit one position to the left. For example,

$$173.246 = 1\times 10^2 + 7\times 10^1 + 3 + 2\times\frac{1}{10} + 4\times\frac{1}{10^2} + 6\times\frac{1}{10^3}$$

There is no reason that a number other than 10 cannot be used as the *base*, or *radix*, of the number system. In fact, bases of 2, 8, and 16 are commonly used in working with digital computers. When the base used is not clear from the context, it is usually indicated as a parenthesized subscript or merely as a subscript. Thus

$$743_{(8)} = 7\times 8^2 + 4\times 8 + 3 = 7\times 64 + 4\times 8 + 3 = 448 + 32 + 3 = 483_{(10)}$$

$$1011.101_{(2)} = 1\times 2^3 + 0\times 2^2 + 1\times 2 + 1 + 1\times\frac{1}{2} + 0\times\frac{1}{4} + 1\times\frac{1}{8} = 11.625_{(10)}$$

Change of Base

In this section it is assumed that all calculations will be performed in base 10, since this is the only base in which most people can easily compute. However, there is no logical reason that some other base could not be used for the computations.

To convert a number from another base into base 10:

Simply write down the digits of the number, with each one multiplied by its appropriate positional value. Then perform the indicated computations in base 10, and write down the answer.

To convert a number from base 10 into another base:

The part of the number to the left of the point and the part to the right must be operated on separately.

For the integer part (the part to the left of the point):

a. Divide the number by the new base, getting an integer quotient and remainder.
b. Write the remainder as the last digit of the number in the new base.
c. Using the quotient from the last division in place of the original number, repeat the above two steps until the quotient becomes zero.

For the fractional part (the part to the right of the point):

a. Multiply the number by the new base.
b. Write down the integral part of the product as the first digit of the fractional part in the new base.
c. Using the fractional part of the last product in place of the original number, repeat the above two steps until the product becomes an integer, or until the desired number of places have been computed.

Examples:

These examples show a convenient method of arranging the computations.

1. *Convert* $103.118_{(10)}$ *to base 8.*

8	103	7
8	12	4
	1	

.118
8
.944
8
7.552
8
4.416
8
3.328
8
2.624
8
4.992

147.074324…

The calculation of the fractional part could be carried out as far as desired. It is a non-terminating fraction that will eventually repeat itself.

$$103.118_{(10)} = 147.074324\ldots_{(8)}$$

The calculations may be further shortened by not writing the multiplier and divisor at each step of the algorithm, as shown in the next example.

Number Systems and Change of Base (continued)

2. *Convert 275.824$_{(10)}$ to base 5.*

5	275	0	.824
	55	0	4.120
	11	1	0.600
	2		3.000

$$275.824_{(10)} = 2100.403_{(5)}$$

To convert from one base to another (neither of which is 10):

The easiest procedure is usually to convert first to base 10, and then to the desired base. However, there are two exceptions to this:

1. If computational facility is possessed in either of the bases, it may be used instead of base 10, and the appropriate one of the above methods applied.
2. If the two bases are different powers of the same number, the conversion may be done digit-by-digit to the base that is the common root of both bases, and then digit-by-digit back to the other base.

Example: Convert 127.653$_{(8)}$ to base 16. (For base 16, the letters A–F are used for the digits 10$_{(10)}$–15$_{(10)}$.)

The first step is to convert the number to base 2, simply by converting each digit to its binary equivalent:

$$127.653_{(8)} = 001\ 010\ 111 \cdot 110\ 101\ 011_{(2)}$$

Now by simply regrouping the binary number into groups of four binary digits, starting at the point, we convert to base 16:

$$127.653_{(16)} = 101\ 0111 \cdot 1101\ 0101\ 1_{(2)} = 57.\text{D}58_{(16)}$$

From Bolz, R.E. and Tuve, G.L., Number systems and logic, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, pp. 949–950.

Binary, Octal, and Decimal Numbers

$10^{\pm n}$ IN OCTAL SCALE

10^n	n	10^{-n}	10^n	n	10^{-n}
1	0	1.000 000 000 000 000	112 402 762 000	10	0.000 000 000 006 676
12	1	0.063 146 314 631 463	1 351 035 564 000	11	0.000 000 000 000 537
144	2	0.005 075 341 217 270	16 432 451 210 000	12	0.000 000 000 000 043
1 750	3	0.000 406 111 564 570	221 441 634 520 000	13	0.000 000 000 000 003
23 420	4	0.000 032 155 613 530	2 657 142 036 440 000	14	0.000 000 000 000 000
303 240	5	0.000 002 476 132 610	34 327 724 461 500 000	15	0.000 000 000 000 000
3 641 100	6	0.000 000 206 157 364	434 157 115 760 200 000	16	0.000 000 000 000 000
46 113 200	7	0.000 000 015 327 745	5 432 127 413 542 400 000	17	0.000 000 000 000 000
575 360 400	8	0.000 000 001 257 143	67 405 553 164 731 000 000	18	0.000 000 000 000 000
7 346 545 000	9	0.000 000 000 104 560			

2^n IN DECIMAL SCALE

n	2^n	n	2^n	n	2^n
0.001	1.00069 33874 62581	0.01	1.00695 55500 56719	0.1	1.07177 34625 36293
0.002	1.00138 72557 11335	0.02	1.01395 94797 90029	0.2	1.14869 83549 97035
0.003	1.00208 16050 79633	0.03	1.02101 21257 07193	0.3	1.23114 44133 44916
0.004	1.00277 64359 01078	0.04	1.02811 38266 56067	0.4	1.31950 79107 72894
0.005	1.00347 17485 09503	0.05	1.03526 49238 41377	0.5	1.41421 35623 73095
0.006	1.00416 75432 38973	0.06	1.04246 57608 41121	0.6	1.51571 65665 10398
0.007	1.00486 38204 23785	0.07	1.04971 66836 23067	0.7	1.62450 47927 12471
0.008	1.00556 05803 98468	0.08	1.05701 80405 61380	0.8	1.74110 11265 92248
0.009	1.00625 78234 97782	0.09	1.06437 01824 53360	0.9	1.86606 59830 73615

$n \log_{10} 2$, $n \log_2 10$ IN DECIMAL SCALE

n	$n \log_{10} 2$	$n \log_2 10$	n	$n \log_{10} 2$	$n \log_2 10$
1	0.30102 99957	3.32192 80949	6	1.80617 99740	19.93156 85693
2	0.60205 99913	6.64385 61898	7	2.10720 99696	23.25349 66642
3	0.90308 99870	9.96578 42847	8	2.40823 99653	26.57542 47591
4	1.20411 99827	13.28771 23795	9	2.70926 99610	29.89735 28540
5	1.50514 99783	16.60964 04744	10	3.01029 99566	33.21928 09489

ADDITION AND MULTIPLICATION TABLES

Binary Scale

Addition	Multiplication
$0 + 0 = 0$	$0 \times 0 = 0$
$0 + 1 = 1 + 0 = 1$	$0 \times 1 = 1 \times 0 = 0$
$1 + 1 = 10$	$1 \times 1 = 1$

Octal Scale

Addition

0	01	02	03	04	05	06	07
1	02	03	04	05	06	07	10
2	03	04	05	06	07	10	11
3	04	05	06	07	10	11	12
4	05	06	07	10	11	12	13
5	06	07	10	11	12	13	14
6	07	10	11	12	13	14	15
7	10	11	12	13	14	15	16

Multiplication

1	02	03	04	05	06	07
2	04	06	10	12	14	16
3	06	11	14	17	22	25
4	10	14	20	24	30	34
5	12	17	24	31	36	43
6	14	22	30	36	44	52
7	16	25	34	43	52	61

Binary, Octal, and Decimal Numbers (continued)

MATHEMATICAL CONSTANTS IN OCTAL SCALE

$\pi =$	$(3.11037\ 555421)_{(8)}$	$e =$	$(2.55760\ 521305)_{(8)}$	$\gamma =$	$(0.44742\ 147707)_{(8)}$
$\pi^{-1} =$	$(0.24276\ 301556)_{(8)}$	$e^{-1} =$	$(0.27426\ 530661)_{(8)}$	$\log_e \gamma =$	$-(0.43127\ 233602)_{(8)}$
$\sqrt{\pi} =$	$(1.61337\ 611067)_{(8)}$	$\sqrt{e} =$	$(1.511411\ 230704)_{(8)}$	$\log_2 \gamma =$	$-(0.62573\ 030645)_{(8)}$
$\log_e \pi =$	$(1.11206\ 404435)_{(8)}$	$\log_{10} e =$	$(0.33626\ 754251)_{(8)}$	$\sqrt{2} =$	$(1.32404\ 746320)_{(8)}$
$\log_2 \pi =$	$(1.51544\ 163223)_{(8)}$	$\log_2 e =$	$(1.34252\ 166245)_{(8)}$	$\log_e 2 =$	$(0.54271\ 027760)_{(8)}$
$\sqrt{10} =$	$(3.12305\ 407267)_{(8)}$	$\log_2 10 =$	$(3.24464\ 741136)_{(8)}$	$\log_e 10 =$	$(2.23273\ 067355)_{(8)}$

From Bolz, R.E. and Tuve, G.L., Number systems and logic, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, p. 951.

Octal-Decimal Integer Conversion

	0	1	2	3	4	5	6	7
0000	0000	0001	0002	0003	0004	0005	0006	0007
0010	0008	0009	0010	0011	0012	0013	0014	0015
0020	0016	0017	0018	0019	0020	0021	0022	0023
0030	0024	0025	0026	0027	0028	0029	0030	0031
0040	0032	0033	0034	0035	0036	0037	0038	0039
0050	0040	0041	0042	0043	0044	0045	0046	0047
0060	0048	0049	0050	0051	0052	0053	0054	0055
0070	0056	0057	0058	0059	0060	0061	0062	0063
1100	0064	0065	0066	0067	0068	0069	0070	0071
0110	0072	0073	0074	0075	0076	0077	0078	0079
0120	0080	0081	0082	0083	0084	0085	0086	0087
0130	0088	0089	0090	0091	0092	0093	0094	0095
0140	0096	0097	0098	0099	0100	0101	0102	0103
0150	0104	0105	0106	0107	0108	0109	0110	0111
0160	0112	0113	0114	0115	0116	0117	0118	0119
0170	0120	0121	0122	0123	0124	0125	0126	0127
0200	0128	0129	0130	0131	0132	0133	0134	0135
0210	0136	0137	0138	0139	0140	0141	0142	0143
0220	0144	0145	0146	0147	0148	0149	0150	0151
0230	0152	0153	0154	0155	0156	0157	0158	0159
0240	0160	0161	0162	0163	0164	0165	0166	0167
0250	0168	0169	0170	0171	0172	0173	0174	0175
0260	0176	0177	0178	0179	0180	0181	0182	0183
0270	0184	0185	0186	0187	0188	0189	0190	0191
0300	0192	0193	0194	0195	0196	0197	0198	0199
0310	0200	0201	0202	0203	0204	0205	0206	0207
0320	0208	0209	0210	0211	0212	0213	0214	0215
0330	0216	0217	0218	0219	0220	0221	0222	0223
0340	0224	0225	0226	0227	0228	0229	0230	0231
0350	0232	0233	0234	0235	0236	0237	0238	0239
0360	0240	0241	0242	0243	0244	0245	0246	0247
0370	0248	0249	0250	0251	0252	0253	0254	0255

	0	1	2	3	4	5	6	7
0400	0256	0257	0258	0259	0260	0261	0262	0263
0410	0264	0265	0266	0267	0268	0269	0270	0271
0420	0272	0273	0274	0275	0276	0277	0278	0279
0430	0280	0281	0282	0283	0284	0285	0286	0287
0440	0288	0289	0290	0291	0292	0293	0294	0295
0450	0296	0297	0298	0299	0300	0301	0302	0303
0460	0304	0305	0306	0307	0308	0309	0310	0311
0470	0312	0313	0314	0315	0316	0317	0318	0319
0500	0320	0321	0322	0323	0324	0325	0326	0327
0510	0328	0329	0330	0331	0332	0333	0334	0335
0520	0336	0337	0338	0339	0340	0341	0342	0343
0530	0344	0345	0346	0347	0348	0349	0350	0351
0540	0352	0353	0354	0355	0356	0357	0358	0359
0550	0360	0361	0362	0363	0364	0365	0366	0367
0560	0368	0369	0370	0371	0372	0373	0374	0375
0570	0376	0377	0378	0379	0380	0381	0382	0383
0600	0384	0385	0386	0387	0388	0389	0390	0391
0610	0392	0393	0394	0395	0396	0397	0398	0399
0620	0400	0401	0402	0403	0404	0405	0406	0407
0630	0408	0409	0410	0411	0412	0413	0414	0415
0640	0416	0417	0418	0419	0420	0421	0422	0423
0650	0424	0425	0426	0427	0428	0429	0430	0431
0660	0432	0433	0434	0435	0436	0437	0438	0439
0670	0440	0441	0442	0443	0444	0445	0446	0447
0700	0448	0449	0450	0451	0452	0453	0454	0455
0710	0456	0457	0458	0459	0460	0461	0462	0463
0720	0464	0465	0466	0467	0468	0469	0470	0471
0730	0472	0473	0474	0475	0476	0477	0478	0479
0740	0480	0481	0482	0483	0484	0485	0486	0487
0750	0488	0489	0490	0491	0492	0493	0494	0495
0760	0496	0497	0498	0499	0500	0501	0502	0503
0770	0504	0505	0506	0507	0508	0509	0510	0511

0000 to 0777 (Octal) — 0000 to 0511 (Decimal)

Octal	Decimal
10000-	4096
20000-	8192
30000-	12288
40000-	16384
50000-	20480
60000-	24576
70000-	28672

	0	1	2	3	4	5	6	7
1000	0512	0513	0514	0515	0516	0517	0518	0519
1010	0520	0521	0522	0523	0524	0525	0526	0527
1020	0528	0529	0530	0531	0532	0533	0534	0535
1030	0536	0537	0538	0539	0540	0541	0542	0543
1040	0544	0545	0546	0547	0548	0549	0550	0551
1050	0552	0553	0554	0555	0556	0557	0558	0559
1060	0560	0561	0562	0563	0564	0565	0566	0567
1070	0568	0569	0570	0571	0572	0573	0574	0575
1100	0576	0577	0578	0579	0580	0581	0582	0583
1110	0584	0585	0586	0587	0588	0589	0590	0591
1120	0592	0593	0594	0595	0596	0597	0598	0599
1130	0600	0601	0602	0603	0604	0605	0606	0607
1140	0608	0609	0610	0611	0612	0613	0614	0615
1150	0616	0617	0618	0619	0620	0621	0622	0623
1160	0624	0625	0626	0627	0628	0629	0630	0631
1170	0632	0633	0634	0635	0636	0637	0638	0639
1200	0640	0641	0642	0643	0644	0645	0646	0647
1210	0648	0649	0650	0651	0652	0653	0654	0655
1220	0656	0657	0658	0659	0660	0661	0662	0663
1230	0664	0665	0666	0667	0668	0669	0670	0671
1240	0672	0673	0674	0675	0676	0677	0678	0679
1250	0680	0681	0682	0683	0684	0685	0686	0687
1260	0688	0689	0690	0691	0692	0693	0694	0695
1270	0696	0697	0698	0699	0700	0701	0702	0703
1300	0704	0705	0706	0707	0708	0709	0710	0711
1310	0712	0713	0714	0715	0716	0717	0718	0719
1320	0720	0721	0722	0723	0724	0725	0726	0727
1330	0728	0729	0730	0731	0732	0733	0734	0735

	0	1	2	3	4	5	6	7
1400	0768	0769	0770	0771	0772	0773	0774	0775
1410	0776	0777	0778	0779	0780	0781	0782	0783
1420	0784	0785	0786	0787	0788	0789	0790	0791
1430	0792	0793	0794	0795	0796	0797	0798	0799
1440	0800	0801	0802	0803	0804	0805	0806	0807
1450	0808	0809	0810	0811	0812	0813	0814	0815
1460	0816	0817	0818	0819	0820	0821	0822	0823
1470	0824	0825	0826	0827	0828	0829	0830	0831
1500	0832	0833	0834	0835	0836	0837	0838	0839
1510	0840	0841	0842	0843	0844	0845	0846	0847
1520	0848	0849	0850	0851	0852	0853	0854	0855
1530	0856	0857	0858	0859	0860	0861	0862	0863
1540	0864	0865	0866	0867	0868	0869	0870	0871
1550	0872	0873	0874	0875	0876	0877	0878	0879
1560	0880	0881	0882	0883	0884	0885	0886	0887
1570	0888	0889	0890	0891	0892	0893	0894	0895
1600	0896	0897	0898	0899	0900	0901	0902	0903
1610	0904	0905	0906	0907	0908	0909	0910	0911
1620	0912	0913	0914	0915	0916	0917	0918	0919
1630	0920	0921	0922	0923	0924	0925	0926	0927
1640	0928	0929	0930	0931	0932	0933	0934	0935
1650	0936	0937	0938	0939	0940	0941	0942	0943
1660	0944	0945	0946	0947	0948	0949	0950	0951
1670	0952	0953	0954	0955	0956	0957	0958	0959
1700	0960	0961	0962	0963	0964	0965	0966	0967
1710	0968	0969	0970	0971	0972	0973	0974	0975
1720	0976	0977	0978	0979	0980	0981	0982	0983
1730	0984	0985	0986	0987	0988	0989	0990	0991

1000 to 1777 (Octal) — 0512 to 1023 (Decimal)

Octal-Decimal Integer Conversion (continued)

1340	0736	0737	0738	0739	0740	0741	0742	0743
1350	0744	0745	0746	0747	0748	0749	0750	0751
1360	0752	0753	0754	0755	0756	0757	0758	0759
1370	0760	0761	0762	0763	0764	0765	0766	0767

1740	0992	0993	0994	0995	0996	0997	0998	0999
1750	1000	1001	1002	1003	1004	1005	1006	1007
1760	1008	1009	1010	1011	1012	1013	1014	1015
1770	1016	1017	1018	1019	1020	1021	1022	1023

2000 to 2777 (Octal) — 1024 to 1535 (Decimal)

Octal	Decimal
10000-	4096
20000-	8192
30000-	12288
40000-	16384
50000-	20480
60000-	24576
70000-	28672

	0	1	2	3	4	5	6	7
2000	1024	1025	1026	1027	1028	1029	1030	1031
2010	1032	1033	1034	1035	1036	1037	1038	1039
2020	1040	1041	1042	1043	1044	1045	1046	1047
2030	1048	1049	1050	1051	1052	1053	1054	1055
2040	1056	1057	1058	1059	1060	1061	1062	1063
2050	1064	1065	1066	1067	1068	1069	1070	1071
2060	1072	1073	1074	1075	1076	1077	1078	1079
2070	1080	1081	1082	1083	1084	1085	1086	1087
2100	1088	1089	1090	1091	1092	1093	1094	1095
2110	1096	1097	1098	1099	1100	1101	1102	1103
2120	1104	1105	1106	1107	1108	1109	1110	1111
2130	1112	1113	1114	1115	1116	1117	1118	1119
2140	1120	1121	1122	1123	1124	1125	1126	1127
2150	1128	1129	1130	1131	1132	1133	1134	1135
2160	1136	1137	1138	1139	1140	1141	1142	1143
2170	1144	1145	1146	1147	1148	1149	1150	1151
2200	1152	1153	1154	1155	1156	1157	1158	1159
2210	1160	1161	1162	1163	1164	1165	1166	1167
2220	1168	1169	1170	1171	1172	1173	1174	1175
2230	1176	1177	1178	1179	1180	1181	1182	1183
2240	1184	1185	1186	1187	1188	1189	1190	1191
2250	1192	1193	1194	1195	1196	1197	1198	1199
2260	1200	1201	1202	1203	1204	1205	1206	1207
2270	1208	1209	1210	1211	1212	1213	1214	1215
2300	1216	1217	1218	1219	1220	1221	1222	1223
2310	1224	1225	1226	1227	1228	1229	1230	1231
2320	1232	1233	1234	1235	1236	1237	1238	1239
2330	1240	1241	1242	1243	1244	1245	1246	1247
2340	1248	1249	1250	1251	1252	1253	1254	1255
2350	1256	1257	1258	1259	1260	1261	1262	1263
2360	1264	1265	1266	1267	1268	1269	1270	1271
2370	1272	1273	1274	1275	1276	1277	1278	1279

	0	1	2	3	4	5	6	7
2400	1280	1281	1282	1283	1284	1285	1286	1287
2410	1288	1289	1290	1291	1292	1293	1294	1295
2420	1296	1297	1298	1299	1300	1301	1302	1303
2430	1304	1305	1306	1307	1308	1309	1310	1311
2440	1312	1313	1314	1315	1316	1317	1318	1319
2450	1320	1321	1322	1323	1324	1325	1326	1327
2460	1328	1329	1330	1331	1332	1333	1334	1335
2470	1336	1337	1338	1339	1340	1341	1342	1343
2500	1344	1345	1346	1347	1348	1349	1350	1351
2510	1352	1353	1354	1355	1356	1357	1358	1359
2520	1360	1361	1362	1363	1364	1365	1366	1367
2530	1368	1369	1370	1371	1372	1373	1374	1375
2540	1376	1377	1378	1379	1380	1381	1382	1383
2550	1384	1385	1386	1387	1388	1389	1390	1391
2560	1392	1393	1394	1395	1396	1397	1398	1399
2570	1400	1401	1402	1403	1404	1405	1406	1407
2600	1408	1409	1410	1411	1412	1413	1414	1415
2610	1416	1417	1418	1419	1420	1421	1422	1423
2620	1424	1425	1426	1427	1428	1429	1430	1431
2630	1432	1433	1434	1435	1436	1437	1438	1439
2640	1440	1441	1442	1443	1444	1445	1446	1447
2650	1448	1449	1450	1451	1452	1453	1454	1455
2660	1456	1457	1458	1459	1460	1461	1462	1463
2670	1464	1465	1466	1467	1468	1469	1470	1471
2700	1472	1473	1474	1475	1476	1477	1478	1479
2710	1480	1481	1482	1483	1484	1485	1486	1487
2720	1488	1489	1490	1491	1492	1493	1494	1495
2730	1496	1497	1498	1499	1500	1501	1502	1503
2740	1504	1505	1506	1507	1508	1509	1510	1511
2750	1512	1513	1514	1515	1516	1517	1518	1519
2760	1520	1521	1522	1523	1524	1525	1526	1527
2770	1528	1529	1530	1531	1532	1533	1534	1535

3000 to 3777 (Octal) — 1536 to 2047 (Decimal)

	0	1	2	3	4	5	6	7
3000	1536	1537	1538	1539	1540	1541	1542	1543
3010	1544	1545	1546	1547	1548	1549	1550	1551
3020	1552	1553	1554	1555	1556	1557	1558	1559
3030	1560	1561	1562	1563	1564	1565	1566	1567
3040	1568	1569	1570	1571	1572	1573	1574	1575
3050	1576	1577	1578	1579	1580	1581	1582	1583
3060	1584	1585	1586	1587	1588	1589	1590	1591
3070	1592	1593	1594	1595	1596	1597	1598	1599
3100	1600	1601	1602	1603	1604	1605	1606	1607
3110	1608	1609	1610	1611	1612	1613	1614	1615
3120	1616	1617	1618	1619	1620	1621	1622	1623
3130	1624	1625	1626	1627	1628	1629	1630	1631
3140	1632	1633	1634	1635	1636	1637	1638	1639
3150	1640	1641	1642	1643	1644	1645	1646	1647
3160	1648	1649	1650	1651	1652	1653	1654	1655
3170	1656	1657	1658	1659	1660	1661	1662	1663
3200	1664	1665	1666	1667	1668	1669	1670	1671
3210	1672	1673	1674	1675	1676	1677	1678	1679
3220	1680	1681	1682	1683	1684	1685	1686	1687
3230	1688	1689	1690	1691	1692	1693	1694	1695
3240	1696	1697	1698	1699	1700	1701	1702	1703
3250	1704	1705	1706	1707	1708	1709	1710	1711
3260	1712	1713	1714	1715	1716	1717	1718	1719
3270	1720	1721	1722	1723	1724	1725	1726	1727
3300	1728	1729	1730	1731	1732	1733	1734	1735

	0	1	2	3	4	5	6	7
3400	1792	1793	1794	1795	1796	1797	1798	1799
3410	1800	1801	1802	1803	1804	1805	1806	1807
3420	1808	1809	1810	1811	1812	1813	1814	1815
3430	1816	1817	1818	1819	1820	1821	1822	1823
3440	1824	1825	1826	1827	1828	1829	1830	1831
3450	1832	1833	1834	1835	1836	1837	1838	1839
3460	1840	1841	1842	1843	1844	1845	1846	1847
3470	1848	1849	1850	1851	1852	1853	1854	1855
3500	1856	1857	1858	1859	1860	1861	1862	1863
3510	1864	1865	1866	1867	1868	1869	1870	1871
3520	1872	1873	1874	1875	1876	1877	1878	1879
3530	1880	1881	1882	1883	1884	1885	1886	1887
3540	1888	1889	1890	1891	1892	1893	1894	1895
3550	1896	1897	1898	1899	1900	1901	1902	1903
3560	1904	1905	1906	1907	1908	1909	1910	1911
3570	1912	1913	1914	1915	1916	1917	1918	1919
3600	1920	1921	1922	1923	1924	1925	1926	1927
3610	1928	1929	1930	1931	1932	1933	1934	1935
3620	1936	1937	1938	1939	1940	1941	1942	1943
3630	1944	1945	1946	1947	1948	1949	1950	1951
3640	1952	1953	1954	1955	1956	1957	1958	1959
3650	1960	1961	1962	1963	1964	1965	1966	1967
3660	1968	1969	1970	1971	1972	1973	1974	1975
3670	1976	1977	1978	1979	1980	1981	1982	1983
3700	1984	1985	1986	1987	1988	1989	1990	1991

Octal-Decimal Integer Conversion (continued)

3310	1736	1737	1738	1739	1740	1741	1742	1743	3710	1992	1993	1994	1995	1996	1997	1998	1999
3320	1744	1745	1746	1747	1748	1749	1750	1751	3720	2000	2001	2002	2003	2004	2005	2006	2007
3330	1752	1753	1754	1755	1756	1757	1758	1759	3730	2008	2009	2010	2011	2012	2013	2014	2015
3340	1760	1761	1762	1763	1764	1765	1766	1767	3740	2016	2017	2018	2019	2020	2021	2022	2023
3350	1768	1769	1770	1771	1772	1773	1774	1775	3750	2024	2025	2026	2027	2028	2029	2030	2031
3360	1776	1777	1778	1779	1780	1781	1782	1783	3760	2032	2033	2034	2035	2036	2037	2038	2039
3370	1784	1785	1786	1787	1788	1789	1790	1791	3770	2040	2041	2042	2043	2044	2045	2046	2047

	0	1	2	3	4	5	6	7		0	1	2	3	4	5	6	7
4000	2048	2049	2050	2051	2052	2053	2054	2055	4400	2304	2305	2306	2307	2308	2309	2310	2311
4010	2056	2057	2058	2059	2060	2061	2062	2063	4410	2312	2313	2314	2315	2316	2317	2318	2319
4020	2064	2065	2066	2067	2068	2069	2070	2071	4420	2320	2321	2322	2323	2324	2325	2326	2327
4030	2072	2073	2074	2075	2076	2077	2078	2079	4430	2328	2329	2330	2331	2332	2333	2334	2335
4040	2080	2081	2082	2083	2084	2085	2086	2087	4440	2336	2337	2338	2339	2340	2341	2343	2343
4050	2088	2089	2090	2091	2092	2093	2094	2095	4450	2344	2345	2346	2347	2348	2349	2350	2351
4060	2096	2097	2098	2099	2100	2101	2102	2103	4460	2352	2353	2354	2355	2356	2357	2358	2359
4070	2104	2105	2106	2107	2108	2109	2110	2111	4470	2360	2361	2362	2363	2364	2365	2366	2367
4100	2112	2113	2114	2115	2116	2117	2118	2119	4500	2368	2369	2370	2371	2372	2373	2374	2375
4110	2120	2121	2122	2123	2124	2125	2126	2127	4510	2376	2377	2378	2379	2380	2381	2382	2383
4120	2128	2129	2130	2131	2132	2133	2134	2135	4520	2384	2385	2386	2387	2388	2389	2390	2391
4130	2136	2137	2138	2139	2140	2141	2142	2143	4530	2392	2393	2394	2395	2396	2397	2398	2399
4140	2144	2145	2146	2147	2148	2149	2150	2151	4540	2400	2401	2402	2403	2404	2405	2406	2407
4150	2152	2153	2154	2155	2156	2157	2158	2159	4550	2408	2409	2410	2411	2412	2413	2414	2415
4160	2160	2161	2162	2163	2164	2165	2166	2167	4560	2416	2417	2418	2419	2420	2421	2422	2423
4170	2168	2169	2170	2171	2172	2173	2174	2175	4570	2424	2425	2426	2427	2428	2429	2430	2431
4200	2176	2177	2178	2179	2180	2181	2182	2183	4600	2432	2433	2434	2435	2436	2437	2438	2439
4210	2184	2185	2186	2187	2188	2189	2190	2191	4610	2440	2441	2442	2443	2444	2445	2446	2447
4220	2192	2193	2194	2195	2196	2197	2198	2199	4620	2448	2449	2450	2451	2452	2453	2454	2455
4230	2200	2201	2202	2203	2204	2205	2206	2207	4630	2456	2457	2458	2459	2460	2461	2462	2463
4240	2208	2209	2210	2211	2212	2213	2214	2215	4640	2464	2465	2466	2467	2468	2469	2470	2471
4250	2216	2217	2218	2219	2220	2221	2222	2223	4650	2472	2473	2474	2475	2476	2477	2478	2479
4260	2224	2225	2226	2227	2228	2229	2230	2231	4660	2480	2481	2482	2483	2484	2485	2486	2487
4270	2232	2233	2234	2235	2236	2237	2238	2239	4670	2488	2489	2490	2491	2492	2493	2494	2495
4300	2240	2241	2242	2243	2244	2245	2246	2247	4700	2496	2497	2498	2499	2500	2501	2502	2503
4310	2248	2249	2250	2251	2252	2253	2254	2255	4710	2504	2505	2506	2507	2508	2509	2510	2511
4320	2256	2257	2258	2259	2260	2261	2262	2263	4720	2512	2513	2514	2515	2516	2517	2518	2519
4330	2264	2265	2266	2267	2268	2269	2270	2271	4730	2520	2521	2522	2523	2524	2525	2526	2527
4340	2272	2273	2274	2275	2276	2277	2278	2279	4740	2528	2529	2530	2531	2532	2533	2534	2535
4350	2280	2281	2282	2283	2284	2285	2286	2287	4750	2536	2537	2538	2539	2540	2541	2542	2543
4360	2288	2289	2290	2291	2292	2293	2294	2295	4760	2544	2545	2546	2547	2548	2549	2550	2551
4370	2296	2297	2298	2299	2300	2301	2302	2303	4770	2552	2553	2554	2555	2556	2557	2558	2559

4000 to 4777 (Octal) 2048 to 2559 (Decimal)

Octal	Decimal
10000-	4096
20000-	8192
30000-	12288
40000-	16384
50000-	20480
60000-	24576
70000-	28672

	0	1	2	3	4	5	6	7		0	1	2	3	4	5	6	7
5000	2560	2561	2562	2563	2564	2565	2566	2567	5400	2816	2817	2818	2819	2820	2821	2822	2823
5010	2568	2569	2570	2571	2572	2572	2574	2575	5410	2824	2825	2826	2827	2828	2829	2830	2831
5020	2576	2577	2578	2579	2580	2581	2582	2583	5420	2832	2833	2834	2835	2836	2837	2838	2839
5030	2584	2585	2586	2587	2588	2589	2590	2591	5430	2840	2841	2842	2843	2844	2845	2846	2847
5040	2592	2593	2594	2595	2596	2597	2598	2599	5440	2848	2849	2850	2851	2852	2853	2854	2855
5050	2600	2601	2602	2603	2604	2605	2606	2607	5450	2856	2857	2858	2859	2860	2861	2862	2863
5060	2608	2609	2610	2611	2612	2613	2614	2615	5460	2864	2865	2866	2867	2868	2869	2870	2871
5070	2616	2617	2618	2619	2620	2621	2622	2623	5470	2872	2873	2874	2875	2876	2877	2878	2879
5100	2624	2625	2626	2627	2628	2629	2630	2631	5500	2880	2881	2882	2883	2884	2885	2886	2887
5110	2632	2633	2634	2635	2636	2637	2638	2639	5510	2888	2889	2890	2891	2892	2893	2894	2895
5120	2640	2641	2642	2643	2644	2645	2646	2647	5520	2896	2897	2898	2899	2900	2901	2902	2903
5130	2648	2649	2650	2651	2652	2653	2654	2655	5530	2904	2905	2906	2907	2908	2909	2910	2911
5140	2656	2657	2658	2659	2660	2661	2662	2663	5540	2912	2913	2914	2915	296	2917	2918	2919
5150	2664	2665	2666	2667	2668	2669	2670	2671	5550	2920	2921	2922	2923	2924	2925	2926	2927
5160	2672	2673	2674	2675	2676	2677	2678	2679	5560	2928	2929	2930	2931	2932	2933	2934	2935
5170	2680	2681	2682	2683	2684	2685	2686	2687	5570	2936	2937	2938	2939	2940	2941	2942	2943
5200	2688	2689	2690	2691	2692	2693	2694	2695	5600	2944	2945	2946	2947	2948	2949	2950	2951
5210	2696	2697	2698	2699	2700	2701	2702	2703	5610	2952	2953	2954	2955	2956	2957	2958	2959
5220	2704	2705	2706	2707	2708	2709	2710	2711	5620	2960	2961	2962	2963	2964	2965	2966	2967
5230	2712	2713	2714	2715	2716	2717	2718	2719	5630	2968	2969	2970	2971	2972	2973	2974	2975
5240	2720	2721	2722	2723	2724	2725	2726	2727	5640	2976	2977	2978	2979	2980	2981	2982	2983

5000 to 5777 (Octal) 2560 to 3071 (Decimal)

Octal-Decimal Integer Conversion (continued)

5250	2728	2729	2730	2731	2732	2733	2734	2735	5650	2984	2985	2986	2987	2988	2989	2990	2991
5260	2736	2737	2738	2739	2740	2741	2742	2743	5660	2992	2993	2994	2995	2996	2997	2998	2999
5270	2744	2745	2746	2747	2748	2749	2750	2751	5670	3000	3001	3002	3003	3004	3005	3006	3007
5300	2752	2753	2754	2755	2756	2757	2758	2759	5700	3008	3009	3010	3011	3012	3013	3014	3015
5310	2760	2761	2762	2763	2764	2765	2766	2767	5710	3016	3017	3018	3019	3020	3021	3022	3023
5320	2768	2769	2770	2771	2772	2773	2774	2775	5720	3024	3025	3026	3027	3028	3029	3030	3031
5330	2776	2777	2778	2779	2780	2781	2782	2783	5730	3032	3033	3034	3035	3036	3037	3038	3039
5340	2784	2785	2786	2787	2788	2789	2790	2791	5740	3040	3041	3042	3043	3044	3045	3046	3047
5350	2792	2793	2794	2795	2796	2797	2798	2799	5750	3048	3049	3050	3051	3052	3053	3054	3055
5360	2800	2801	2802	2803	2804	2805	2806	2807	5760	3056	3057	3058	3059	3060	3061	3062	3063
5370	2808	2809	2810	2811	2812	2813	2814	2815	5770	3064	3065	3066	3067	3068	3069	3070	3071

6000 to 6777 (Octal) — 3072 to 3583 (Decimal)

Octal	Decimal
10000-	4096
20000-	8192
30000-	12288
40000-	16384
50000-	20480
60000-	24576
70000-	28672

	0	1	2	3	4	5	6	7		0	1	2	3	4	5	6	7
6000	3072	3073	3074	3075	3076	3077	3078	3079	6400	3328	3329	3330	3331	3332	3333	3334	3335
6010	3080	3081	3082	3083	3084	3085	3086	3087	6410	3336	3337	3338	3339	3340	3341	3342	3343
6020	3088	3089	3090	3091	3092	3093	3094	3095	6420	3344	3345	3346	3347	3348	3349	3350	3351
6030	3096	3097	3098	3099	3100	3101	3102	3103	6430	3352	3353	3354	3355	3356	3357	3358	3359
6040	3104	3105	3106	3107	3108	3109	3110	3111	6440	3360	3361	3362	3363	3364	3365	3366	3367
6050	3112	3113	3114	3115	3116	3117	3118	3119	6450	3368	3369	3370	3371	3372	3373	3374	3375
6060	3120	3121	3122	3123	3124	3125	3126	3127	6460	3376	3377	3378	3379	3380	3381	3382	3383
6070	3128	3129	3139	3131	3132	3133	3134	3135	6470	3384	3385	3386	3387	3388	3389	3390	3391
6100	3136	3137	3138	3139	3140	3141	3142	3143	6500	3392	3393	3394	3395	3396	3397	3398	3399
6110	3144	3145	3146	3147	3148	3149	3150	3151	6510	3400	3401	3402	3403	3404	3405	3406	3407
6120	3152	3153	3154	3155	3156	3157	3158	3159	6520	3408	3409	3410	3411	3412	3413	3414	3415
6130	3160	3161	3162	3163	3164	3165	3166	3167	6530	3416	3417	3418	3419	3420	3421	3422	3423
6140	3168	3169	3170	3171	3172	3173	3174	3175	6540	3424	3425	3426	3427	3428	3429	3430	3431
6150	3176	3177	3178	3179	3180	3181	3182	3183	6550	3432	3433	3434	3435	3436	3437	3438	3439
6160	3184	3185	3186	3187	3188	3189	3190	3191	6560	3440	3441	3442	3443	3444	3445	3446	3447
6170	3192	3193	3194	3195	3196	3197	3198	3199	6570	3448	3449	3450	3451	3452	3453	3454	3455
6200	3200	3201	3202	3203	3204	3205	3206	3207	6600	3456	3457	3458	3459	3460	3461	3462	3463
6210	3208	3209	3210	3211	3212	3213	3214	3215	6610	3464	3465	3466	3467	3468	3469	3470	3471
6220	3216	3217	3218	3219	3220	3221	3222	3223	6620	3472	3473	3474	3475	3476	3477	3478	3479
6230	3224	3225	3226	3227	3228	3229	3230	3231	6630	3480	3481	3482	3483	3484	3485	3486	3487
6240	3232	3233	3234	3235	3236	3237	3238	3239	6640	3488	3489	3490	3491	3492	3493	3494	3495
6250	3240	3241	3242	3243	3244	3245	3246	3247	6650	3496	3497	3498	3499	3500	3501	3502	3503
6260	3248	3249	3250	3251	3252	3253	3254	3255	6660	3504	3505	3506	3507	3508	3509	3510	3511
6270	3256	3267	3258	3259	3260	3261	3262	3263	6670	3512	3513	3514	3515	3516	3517	3518	3519
6300	3264	3265	3266	3267	3268	3269	3270	3271	6700	3520	3521	3522	3523	3524	3525	3526	3527
6310	3272	3273	3274	3275	3276	3277	3278	3279	6710	3528	3529	3530	3531	3532	3533	3534	3535
6320	3280	3281	3282	3283	3284	3285	3286	3287	6720	3536	3537	3538	3539	3540	3541	3542	3543
6330	3288	3289	3290	3291	3292	3293	3294	3295	6730	3544	3545	3546	3547	3548	3549	3550	3551
6340	3296	3297	3298	3299	3300	3301	3302	3303	6740	3552	3553	3554	3555	3556	3557	3558	3559
6350	3304	3305	3306	3307	3308	3309	3310	3311	6750	3560	3561	3562	3563	3564	3565	3566	3567
6360	3312	3313	3314	3315	3316	3317	3318	3319	6760	3568	3569	3570	3571	3572	3573	3574	3575
6370	3320	3321	3322	3323	3324	3325	3326	3327	6770	3576	3577	3578	3579	3580	3581	3582	3583

7000 to 7777 (Octal) — 3584 to 4095 (Decimal)

	0	1	2	3	4	5	6	7		0	1	2	3	4	5	6	7
7000	3584	3585	3586	3587	3588	3589	3590	3591	7400	3840	3841	3842	3843	3844	3845	3846	3847
7010	3592	3593	3594	3595	3596	3597	3598	3599	7410	3848	3849	3850	3851	3852	3853	3854	3855
7020	3600	3601	3602	3603	3604	3605	3606	3607	7420	3856	3857	3858	3859	3860	3861	3862	3863
7030	3608	3609	3610	3611	3612	3613	3614	3615	7430	3864	3865	3866	3867	3868	3869	3870	3871
7040	3616	3617	3618	3619	3620	3621	3622	3623	7440	3872	3873	3874	3875	3876	3877	3878	3879
7050	3624	3625	3626	3627	3628	3629	3630	3631	7450	3880	3881	3882	3883	3884	3885	3886	3887
7060	3632	3633	3634	3635	3636	3637	3638	3639	7460	3888	3889	3890	3891	3892	3893	3894	3895
7070	3640	3641	3642	3643	3644	3645	3646	3647	7470	3896	3897	3898	3899	3900	3901	3902	3903
7100	3648	3649	3650	3651	3652	3653	3654	3655	7500	3904	3905	3906	3907	3908	3909	3910	3911
7110	3656	3657	3658	3659	3660	3661	3662	3663	7510	3912	3913	3914	3915	3916	3917	3918	3919
7120	3664	3665	3666	3667	3668	3669	3670	3671	7520	3920	3921	3922	3923	3924	3925	3926	3927
7130	3672	3673	3674	3675	3676	3677	3678	3679	7530	3928	3929	3930	3931	3932	3933	3934	3935
7140	3680	3681	3682	3683	3684	3685	3686	3687	7540	3936	3937	3938	3939	3940	3941	3942	3943
7150	3688	3689	3690	3691	3692	3693	3694	3695	7550	3944	3945	3946	3947	3948	3949	3950	3951
7160	3696	3697	3698	3699	3700	3701	3702	3703	7560	3952	3953	3954	3955	3956	3957	3958	3959
7170	3704	3705	3706	3707	3708	3709	3710	3711	7570	3960	3961	3962	3963	3964	3965	3966	3967

Octal-Decimal Integer Conversion (continued)

7200	3712	3713	3714	3715	3716	3717	3718	3719	7600	3968	3969	3970	3971	3972	3973	3974	3975
7210	3720	3721	3722	3723	3724	3725	3726	3727	7610	3976	3977	3978	3979	3980	3981	3982	3983
7220	3728	3729	3730	3731	3732	3733	3734	3735	7620	3984	3985	3986	3987	3988	3989	3990	3991
7230	3736	3737	3738	3739	3740	3741	3742	3743	7630	3992	3993	3994	3995	3996	3997	3998	3999
7240	3744	3745	3746	3747	3748	3749	3750	3751	7640	4000	4001	4002	4003	4004	4005	4006	4007
7250	3752	3753	3754	3755	3756	3757	3758	3759	7650	4008	4009	4010	4011	4012	4013	4014	4015
7260	3760	3761	3762	3763	3764	3765	3766	3767	7660	4016	4017	4018	4019	4020	4021	4022	4023
7270	3768	3769	3770	3771	3772	3773	3774	3775	7670	4024	4025	4026	4027	4028	4029	4030	4031
7300	3776	3777	3778	3779	3780	3781	3782	3783	7700	4032	4033	4034	4035	4036	4037	4038	4039
7310	3784	3785	3786	3787	3788	3789	3790	3791	7710	4040	4041	4042	4043	4044	4045	4046	4047
7320	3792	3793	3794	3795	3796	3797	3798	3799	7720	4048	4049	4050	4051	4052	4053	4054	4055
7330	3800	3801	3802	3803	3804	3805	3806	3807	7730	4056	4057	4058	4059	4060	4061	4062	4063
7340	3808	3809	3810	3811	3812	3813	3814	3815	7740	4064	4065	4066	4067	4068	4069	4070	4071
7350	3816	3817	3818	3819	3820	3821	3822	3823	7750	4072	4073	4074	4075	4076	4077	4078	4079
7360	3824	3825	3826	3827	3828	3829	3830	3831	7760	4080	4081	4082	4083	4084	4085	4086	4087
7360	3832	3833	3834	3835	3836	3837	3838	3839	7770	4088	4089	4090	4091	4092	4093	4094	4095

From Bolz, R.E. and Tuve, G.L., Number systems and logic, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, pp. 952–955.

Boolean Theorems

$$A + 0 = A \qquad \overline{AB} = \overline{A} + \overline{B}$$

$$A \cdot 1 = A \qquad (A + B) + C = A + (B + C)$$

$$A + A = A \qquad (AB)C = A(BC)$$

$$A \cdot A = A \qquad A + \overline{A}B = A + B$$

$$A + 1 = 1 \qquad A\left(\overline{A} + B\right) = AB$$

$$A \cdot 0 = 0 \qquad \left(A + B\right)\left(\overline{A} + C\right) = AC + \overline{A}B$$

$$A + AB = A \qquad \left(\overline{AC + BC}\right) = \overline{A}C + \overline{B}\overline{C}$$

$$\overline{\overline{A}} = A \qquad \left(A + C\right)\left(B + \overline{C}\right) = \left(\overline{A} + C\right)\left(\overline{B} + \overline{C}\right)$$

$$\overline{A + B} = \overline{A}\,\overline{B}$$

EXPLANATION:

These Boolean theorems (sometimes called switching theorems) are used in problems involving binary states. The two states may be considered as functional propositions, true or false (hence the alternate name "propositional calculus"). But in physical devices, such as switches, controls, or computers, the two states may be on or off, short circuit or open circuit, high voltage or low voltage, or presence or absence of a hole in a card or tape, and the digits 1 and 0 are arbitrarily used.

In these theorems each of the variables can represent an arbitrary function. One method for manipulating forms in switching algebra is to use a map.

Since the use of symbols in Boolean algebra has not yet been fully standardized, the following is a detailed explanation of the symbols used in the above table.

SYMBOLS:

A, B, and C are variables.

The bar above the variable indicates the negation of the variable, e.g., $\overline{\text{A}}$ means "not A".

The plus sign, +, is used for the *or* function. This function does not obey the conventional arithmetical rules for sums.

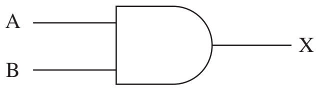
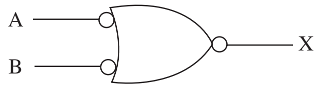
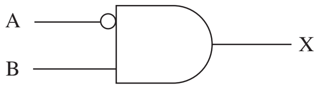
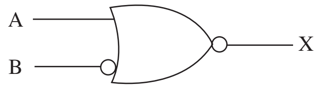
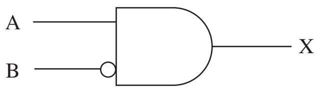
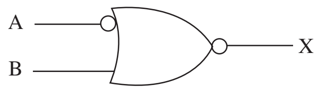
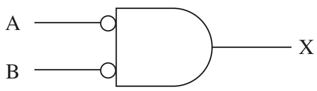
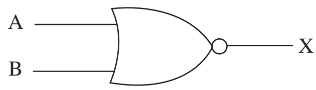
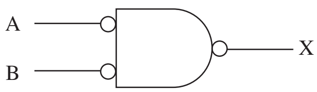
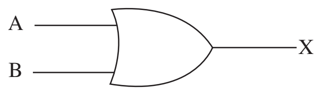
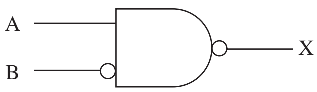
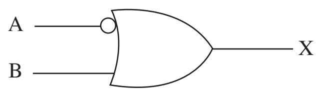
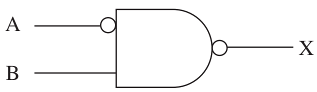
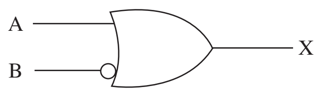
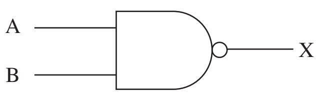
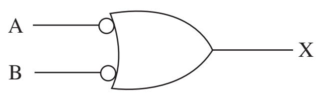
The multiplication sign, ·, is used for the *and* function, sometimes called conjunction. This function obeys the conventional arithmetical rules for products. Thus if the binary values are taken arithmetically as one and zero, 1·1 = 1, and 1·0 = 0. But, in Boolean notation, 1 + 1 = 1, which is not correct by arithmetical notation.

If a variable (e.g., a switch) can have only two sates, designated as 1 or 0, it follows that $\overline{1}$ is equivalent to 0, and $\overline{0}$ is equivalent to 1.

From Bolz, R.E. and Tuve, G.L., Number systems and logic, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, p. 967.

Applications and Functions of Two Variables

Table of Combinations

AND	OR	A	B	X
		H H L L	H L H L	H L L L
		H H L L	H L H L	L L H L
		H H L L	H L H L	L H L L
		H H L L	H L H L	L L L H
		H H L L	H L H L	H H H L
		H H L L	H L H L	H L H H
		H H L L	H L H L	H H L H
		H H L L	H L H L	L H H H

From Bolz, R.E. and Tuve, G.L., Number systems and logic, in *CRC Handbook of Tables for Applied Engineering Science*, CRC Press, Boca Raton, FL, 1973, p. 968, Originally from MIL-STD 806B, February 1962.