CHAPTER 22
TRANSMISSION LINES

GENERAL

The equations and charts of this chapter are for transmission lines operating in the TEM mode.* At the beginning of several of the sections (e.g., "Fundamental Quantities," "Voltage and Current," "Impedance and Admittance," "Reflection Coefficient") there are accurate equations, according to conventional transmission-line theory. These are applicable from the lowest power and communication frequencies, including direct current, up to the frequency where a higher mode begins to appear on the line.

Following the accurate equations are others that are specially adapted for use in radio-frequency problems. In cases of small attenuation, the terms $\alpha^2 x^2$ and higher powers in the expansion of $\exp \alpha x$, etc., are neglected. Thus, when $\alpha x = (\alpha/\beta)\theta = 0.1$ neper (or about 1 decibel), the error in the approximate equations is of the order of 1 percent.

Much of the information is useful also in connection with special lines, such as those with spiral (helical) inner conductors, which function in a quasi-TEM mode; likewise for microstrip.

It should be observed that Z_0 and Y_0 are complex quantities and the imaginary part cannot be neglected in the accurate equations, unless preliminary examination of the problems indicates the contrary. Even when attenuation is small, $Z_0 = 1/Y_0$ must often be taken at its complex value, especially when the standing-wave ratio is high. In the first few pages of equations, the symbol R_0 is used frequently. However, in later charts and special applications, the conventional symbol Z_0 is used where the context indicates that the quadrature component need not be considered for the moment.

Rule of Subscripts and Sign Conventions

The equations for voltage, impedance, etc., are generally for the quantities at the input terminals of the line in terms of those at the output terminals (Fig. 1). In case it is desired to find the quantities at the output in terms of those at the input, it is simply necessary to interchange the subscripts 1 and 2 in the equations and to place a minus sign before x or θ. The minus sign may then be cleared through the hyperbolic or circular functions; thus

$$\sinh(-\gamma x) = -\sinh\gamma x, \quad \text{etc.}$$

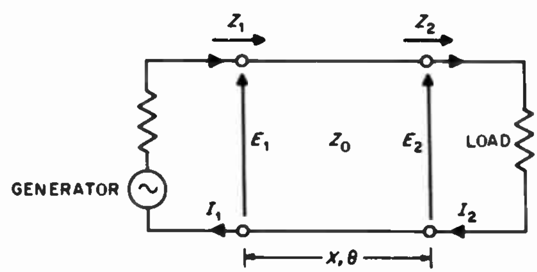

Fig. 1—Transmission line with generator, load.

* The information on pp. 22-1–22-18 is valid for single-mode waveguides in general, except for equations where the symbols R, L, G, or C per unit length are involved.

SYMBOLS

Voltage and current symbols usually represent the alternating-current complex sinusoid, with magnitude equal to the root-mean-square value of the quantity.

Certain quantities, namely C, c, f, L, T, v, and ω are shown with an optional set of units in parentheses. Either the standard units or the optional units may be used, provided the same set is used throughout.

$A = 10 \log_{10}(1/\eta)$ = dissipation loss in a length of line in decibels

$A_0 = 8.686\alpha x$ = normal or matched-line attenuation of a length of line in decibels.

B_0 = susceptive component of Y_0 in mhos

C = capacitance of line in farads/unit length (microfarads/unit length)

c = velocity of light in vacuum in units of length/second (units of length/microsecond). See page 3-11.

E = voltage (root-mean-square complex sinusoid) in volts

${}_fE$ = voltage of forward wave, traveling toward load

${}_rE$ = voltage of reflected wave

$|E_{flat}|$ = root-mean-square voltage when standing-wave ratio = 1.0

$|E_{max}|$ = root-mean-square voltage at crest of standing wave

$|E_{min}|$ = root-mean-square voltage at trough of standing wave

e = instantaneous voltage

$F_p = G/\omega C$ = power factor of dielectric

f = frequency in hertz (megahertz)

G = conductance of line in mhos/unit length

G_0 = conductive component of Y_0 in mhos

$g_a = Y_a/Y_0$ = normalized admittance at voltage standing-wave maximum

$g_b = Y_b/Y_0$ = normalized admittance at voltage standing-wave minimum

I = current (root-mean-square complex sinusoid) in amperes

${}_fI$ = current of forward wave, traveling toward load

${}_rI$ = current of reflected wave

i = instantaneous current

L = inductance of line in henries/unit length (microhenries/unit length)

P = power in watts

R = resistance of line in ohms/unit length

R_0 = resistive component of Z_0 in ohms

$r_a = Z_a/Z_0$ = normalized impedance at voltage standing-wave maximum

$r_b = Z_b/Z_0$ = normalized impedance at voltage standing-wave minimum

$S = |E_{max}/E_{min}|$ = voltage standing-wave ratio

T = delay of line in seconds/unit length (microseconds/unit length)

v = phase velocity of propagation in units of length/second (units of length/microsecond)

X_0 = reactive component of Z_0 in ohms

x = distance between points 1 and 2 in units of length (also used for normalized reactance $= X/Z_0$)

$Y_1 = G_1 + jB_1 = 1/Z_1$ = admittance in mhos looking toward load from point 1

$Y_0 = G_0 + jB_0 = 1/Z_0$ = characteristic admittance of line in mhos

$Z_1 = R_1 + jX_1$ = impedance in ohms looking toward load from point 1

$Z_0 = R_0 + jX_0$ = characteristic impedance of line in ohms

Z_{oc} = input impedance of a line open-circuited at the far end

Z_{sc} = input impedance of a line short-circuited at the far end

α = attenuation constant = nepers/unit length = 0.1151 × decibels/unit length

β = phase constant in radians/unit length

$\gamma = \alpha + j\beta$ = propagation constant

ϵ = base of natural logarithms = 2.718; or dielectric constant of medium (relative to air), according to context

$\eta = P_2/P_1$ = efficiency (fractional)

$\theta = \beta x$ = electrical length or angle of line in radians

$\theta^\circ = 57.3\theta$ = electrical angle of line in degrees

λ = wavelength in units of length

λ_0 = wavelength in free space

$\rho = |\rho| \angle 2\psi$ = voltage reflection coefficient

$\rho_{dB} = -20 \log_{10}(1/\rho)$ = voltage reflection coefficient in decibels

ϕ = time phase angle of complex voltage at voltage standing-wave maximum

ψ = half the angle of the reflection coefficient = electrical angle to nearest voltage standing-wave maximum on the generator side

$\omega = 2\pi f$ = angular velocity in radians/second (radians/microsecond).

FUNDAMENTAL QUANTITIES AND LINE PARAMETERS

$$dE/dx = -(R + j\omega L) I$$

$$d^2E/dx^2 = \gamma^2 E$$

$$dI/dx = -(G + j\omega C) E$$

$$d^2I/dx^2 = \gamma^2 I$$

$$\gamma = \alpha + j\beta = [(R + j\omega L)(G + j\omega C)]^{1/2}$$

$$= j\omega (LC)^{1/2} \times [(1 - jR/\omega L)(1 - jG/\omega C)]^{1/2}$$

$$\alpha = (\tfrac{1}{2}\{[(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)]^{1/2} + RG - \omega^2 LC\})^{1/2}$$

$$\beta = (\tfrac{1}{2}\{[(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)]^{1/2} - RG + \omega^2 LC\})^{1/2}$$

$$Z_0 = 1/Y_0 = [(R + j\omega L)/(G + j\omega C)]^{1/2}$$

$$= (L/C)^{1/2} [(1 - jR/\omega L)/(1 - jG/\omega C]^{1/2}$$

$$= R_0(1 + jX_0/R_0)$$

$$Y_0 = 1/Z_0 = G_0(1 + j\, B_0/G_0)$$

$$\alpha = \tfrac{1}{2}(R/R_0 + G/G_0)$$

$$\beta\, B_0/G_0 = \tfrac{1}{2}(R/R_0 - G/G_0)$$

$$R_0 = [M/2(G^2 + \omega^2 C^2)]^{1/2}$$

$$G_0 = [M/2(R^2 + \omega^2 L^2)]^{1/2}$$

$$B_0/G_0 = -X_0/R_0 = (\omega RC - \omega LG)/M$$

where

$$M=[(R^2+\omega^2L^2)(G^2+\omega^2C^2)]^{1/2}+RG+\omega^2LC$$

$$1/T=v=f\lambda=\omega/\beta$$

$$\beta=\omega/v=\omega T=2\pi/\lambda$$

$$\gamma x=\alpha x+j\beta x=(\alpha/\beta)\theta+j\theta$$

$$\theta=\beta x=2\pi\, x/\lambda=2\pi\, fTx$$

$$\theta^\circ=57.3\theta=360x/\lambda=360fTx.$$

(A) Special case—distortionless line: When $R/L=G/C$, the quantities Z_0 and α are independent of frequency.

$$X_0=0$$

$$\alpha=R/R_0$$

$$Z_0=R_0+j0=(L/C)^{1/2}$$

$$\beta=\omega(LC)^{1/2}.$$

(B) For small attenuation: $R/\omega L$ and $G/\omega C$ are small.

$$\gamma=j\omega(LC)^{1/2}\{1-j[(R/2\omega L)+(G/2\omega C)]\}$$

$$=j\beta(1-j\alpha/\beta)$$

$$\beta=\omega(LC)^{1/2}=\omega L/R_0=\omega CR_0$$

$$T=1/v=(LC)^{1/2}=R_0C$$

$$\alpha/\beta=(R/2\omega L)+(G/2\omega C)=(R/2\omega L)+\tfrac{1}{2}F_p$$

$$=(Rv/2\omega R_0)+\tfrac{1}{2}F_p$$

= attenuation in nepers/radian

$$=\frac{\text{(decibels per 100 feet) (wavelength in line in meters)}}{1663}$$

$$\alpha=\tfrac{1}{2}R(C/L)^{1/2}+\tfrac{1}{2}G(L/C)^{1/2}$$

$$=(R/2R_0)+\pi(F_p/\lambda)$$

$$=(R/2R_0)+\tfrac{1}{2}(F_p\beta)$$

where R and G vary with frequency, while L and C are nearly independent of frequency.

$$Z_0=1/Y_0$$

$$=(L/C)^{1/2}\{1-j[(R/2\omega L)-(G/2\omega C)]\}$$

$$=R_0(1+jX_0/R_0)$$

$$=1/[G_0(1+j\,B_0/G_0)]$$

$$=(1/G_0)(1-jB_0/G_0)$$

$$R_0=1/G_0=(L/C)^{1/2}$$

$$B_0/G_0=-(X_0/R_0)=(R/2\omega L)-\tfrac{1}{2}F_p=(\alpha/\beta)-F_p$$

$$X_0=-[R/2\omega(LC)^{1/2}]+(G/2\omega C)(L/C)^{1/2}$$

$$=-(R\lambda/4\pi)+(\tfrac{1}{2}F_p)R_0.$$

(C) With certain exceptions, the following few equations are for ordinary lines (e.g., not spiral delay lines) with the field totally immersed in a uniform dielectric of dielectric constant ϵ (relative to air). The exceptions are all the quantities not including the symbol ϵ, these being good also for special types such as spiral delay lines, microstrip, etc.

$L=1.016R_0(\epsilon^{1/2})\times10^{-3}$ microhenries/foot

$=\tfrac{1}{3}R_0(\epsilon^{1/2})\times10^{-4}$ microhenries/centimeter

$C=1.016[(\epsilon^{1/2})/R_0]\times10^{-3}$ microfarads/foot

$=[(\epsilon^{1/2})/3R_0]\times10^{-4}$ microfarads/centimeter

$v/c=1016/R_0C'=\epsilon^{-1/2}$

= velocity factor (with capacitance C' in picofarads/foot)

$\lambda=\lambda_0v/c=c/f(\epsilon^{1/2})=\lambda_0/(\epsilon^{1/2})$

$T=1/v=R_0C'\times10^{-6}=1.016\times10^{-3}/(v/c)$

$=1.016\times10^{-3}\epsilon^{1/2}$ microseconds/foot (with capacitance C' in picofarads/foot).

The line length is

$x/\lambda=xf(\epsilon^{1/2})/984$ wavelengths

$\theta=2\pi x/\lambda=xf(\epsilon^{1/2})/156.5$ radians

where xf is the product of feet times megahertz.

VOLTAGE AND CURRENT

$$E_1={}_fE_1+{}_rE_1={}_fE_2\epsilon^{\gamma x}+{}_rE_2\epsilon^{-\gamma x}$$

$$=E_2\{[(Z_2+Z_0)/2Z_2]\epsilon^{\gamma x}+[(Z_2-Z_0)/2Z_2]\epsilon^{-\gamma x}\}$$

$$=\tfrac{1}{2}(E_2+I_2Z_0)\epsilon^{\gamma x}+\tfrac{1}{2}(E_2-I_2Z_0)\epsilon^{-\gamma x}$$

$$=E_2[\cosh\gamma x+(Z_0/Z_2)\sinh\gamma x]$$

$$=E_2\cosh\gamma x+I_2Z_0\sinh\gamma x$$

$$=[E_2/(1+\rho_2)](\epsilon^{\gamma x}+\rho_2\epsilon^{-\gamma x})$$

$$I_1={}_fI_1+{}_rI_1={}_fI_2\epsilon^{\gamma x}+{}_rI_2\epsilon^{-\gamma x}$$

$$=Y_0({}_fE_2\epsilon^{\gamma x}-{}_rE_2\epsilon^{-\gamma x})$$

$$=I_2\{[(Z_0-Z_2)/2Z_0]\epsilon^{\gamma x}+[(Z_0+Z_2)/2Z_0]\epsilon^{-\gamma x}\}$$

$$=\tfrac{1}{2}(I_2+E_2Y_0)\epsilon^{\gamma x}+\tfrac{1}{2}(I_2-E_2Y_0)\epsilon^{-\gamma x}$$

$$=I_2[\cosh\gamma x+(Z_2/Z_0)\sinh\gamma x]$$

$$=I_2\cosh\gamma x+E_2Y_0\sinh\gamma x$$

$$=[I_2/(1-\rho_2)](\epsilon^{\gamma x}-\rho_2\epsilon^{-\gamma x})$$

$$E_1=AE_2+BI_2$$

$$I_1=CE_2+DI_2$$

where the general circuit parameters are $A=\cosh\gamma x$, $B=Z_0\sinh\gamma x$, $C=Y_0\sinh\gamma x$, and $D=\cosh\gamma x$.

Refer to section on "Matrix Algebra."

(A) When point 2 is at a voltage maximum or minimum, x' is measured from voltage maximum and x'' from voltage minimum (similarly for currents)

$$E_1=E_{max}(\cosh\gamma x'+S^{-1}\sinh\gamma x')$$
$$=E_{min}(\cosh\gamma x''+S\sinh\gamma x'')$$
$$I_1=I_{max}(\cosh\gamma x'+S^{-1}\sinh\gamma x')$$
$$=I_{min}(\cosh\gamma x''+S\sinh\gamma x'').$$

When attenuation is neglected

$$E_1=E_{max}(\cos\theta'+jS^{-1}\sin\theta')$$
$$=E_{min}(\cos\theta''+jS\sin\theta'').$$

(B) Letting Z_l= impedance of load, l= distance from load to point 2, and x_l= distance from load to point 1

$$E_1=E_2\frac{\cosh\gamma x_l+(Z_0/Z_l)\sinh\gamma x_l}{\cosh\gamma l+(Z_0/Z_l)\sinh\gamma l}$$

$$I_1=I_2\frac{\cosh\gamma x_l+(Z_l/Z_0)\sinh\gamma x_l}{\cosh\gamma l+(Z_l/Z_0)\sinh\gamma l}.$$

(C) $$e_1=\sqrt{2}\,|{}_fE_2|\,\epsilon^{\alpha x}\sin[\omega t+2\pi(x/\lambda)-\psi_2+\phi]$$
$$+\sqrt{2}\,|{}_rE_2|\,\epsilon^{-\alpha x}\sin[\omega t-2\pi(x/\lambda)+\psi_2+\phi]$$
$$i_1=\sqrt{2}\,|{}_fI_2|\,\epsilon^{\alpha x}$$
$$\times\sin[\omega t+2\pi(x/\lambda)-\psi_2+\phi+\tan^{-1}(B_0/G_0)]$$
$$+\sqrt{2}\,|{}_rI_2|\,\epsilon^{-\alpha x}$$
$$\times\sin[\omega t-2\pi(x/\lambda)+\psi_2+\phi+\tan^{-1}(B_0/G_0)].$$

(D) For small attenuation

$$E_1=E_2\{[1+(Z_0/Z_2)\alpha x]\cos\theta$$
$$+j[(Z_0/Z_2)+\alpha x]\sin\theta\}$$
$$I_1=I_2\{[1+(Z_2/Z_0)\alpha x]\cos\theta$$
$$+j[(Z_2/Z_0)+\alpha x]\sin\theta\}.$$

(E) When attenuation is neglected

$$E_1=E_2\cos\theta+jI_2Z_0\sin\theta$$
$$=E_2[\cos\theta+j(Y_2/Y_0)\sin\theta]$$
$$={}_fE_2\epsilon^{j\theta}+{}_rE_2\epsilon^{-j\theta}$$
$$I_1=I_2\cos\theta+jE_2Y_0\sin\theta$$
$$=I_2[\cos\theta+j(Z_2/Z_0)\sin\theta]$$
$$=Y_0({}_fE_2\epsilon^{j\theta}-{}_rE_2\epsilon^{-j\theta}).$$

General circuit parameters are

$$A=\cos\theta$$
$$B=jZ_0\sin\theta$$
$$C=jY_0\sin\theta$$
$$D=\cos\theta.$$

IMPEDANCE AND ADMITTANCE

$$\frac{Z_1}{Z_0}=\frac{Z_2\cosh\gamma x+Z_0\sinh\gamma x}{Z_0\cosh\gamma x+Z_2\sinh\gamma x}$$

$$\frac{Y_1}{Y_0}=\frac{Y_2\cosh\gamma x+Y_0\sinh\gamma x}{Y_0\cosh\gamma x+Y_2\sinh\gamma x}.$$

(A) By interchange of subscripts and change of signs (see p. 22-1), the load impedance is

$$\frac{Z_2}{Z_0}=\frac{Z_1\cosh\gamma x-Z_0\sinh\gamma x}{Z_0\cosh\gamma x-Z_1\sinh\gamma x}.$$

For a length of uniform line or a symmetrical network

$$Z_1=Z_{oc}(Z_2+Z_{sc})/(Z_2+Z_{oc})$$
$$Z_2=Z_{oc}(Z_1-Z_{sc})/(Z_{oc}-Z_1).$$

(B) The input impedance of a line at a position of maximum or minimum voltage has the same phase angle as the characteristic impedance.

$$Z_1/Z_0=Z_b/Z_0=Y_0/Y_b=r_b+j0=S^{-1}$$

at a voltage minimum (current maximum).

$$Y_1/Y_0=Y_a/Y_0=Z_0/Z_a=g_a+j0=S^{-1}$$

at a voltage maximum (current minimum).

(C) When attenuation is small

$$\frac{Z_1}{Z_0}=\frac{[(Z_2/Z_0)+\alpha x]+j[1+(Z_2/Z_0)\alpha x]\tan\theta}{[1+(Z_2/Z_0)\alpha x]+j[(Z_2/Z_0)+\alpha x]\tan\theta}.$$

For admittances, replace Z_0, Z_1, and Z_2 by Y_0, Y_1, and Y_2, respectively.

When A and B are real

$$\frac{A\pm jB\tan\theta}{B\pm jA\tan\theta}=\frac{2AB\pm j(B^2-A^2)\sin 2\theta}{(B^2+A^2)+(B^2-A^2)\cos 2\theta}.$$

(D) When attenuation is neglected

$$\frac{Z_1}{Z_0}=\frac{Z_2/Z_0+j\tan\theta}{1+j(Z_2/Z_0)\tan\theta}=\frac{1-j(Z_2/Z_0)\cot\theta}{Z_2/Z_0-j\cot\theta}$$

and similarly for admittances.

(E) When attenuation $\alpha x=\theta\alpha/\beta$ is small and standing-wave ratio is large (say >10) (*Note*: The complex value of Z_0 or Y_0 must be used in computing the resistive component of Z_1 or Y_1.